

The S-matrix of the $\text{AdS}_5 \times \text{S}^5$ superstring

Marius de Leeuw¹

*Institute for Theoretical Physics and Spinoza Institute,
Utrecht University, 3508 TD Utrecht, The Netherlands*

Abstract

In this article we review the world-sheet scattering theory of strings on $\text{AdS}_5 \times \text{S}^5$. The asymptotic spectrum of this world-sheet theory contains both fundamental particles and bound states of the latter. We explicitly derive the S-matrix that describes scattering of arbitrary bound states. The key feature that enables this derivation is the so-called Yangian symmetry which is related to the centrally extended $\mathfrak{su}(2|2)$ superalgebra. Subsequently, we study the universal algebraic properties of the found S-matrix. As in many integrable models, the S-matrix plays a key role in the determination of the energy spectrum. In this context, we employ the Bethe ansatz approach to compute the large volume energy spectrum of string bound states.

¹e-mail: M.deLeeuw@uu.nl

Contents

1	Introduction	7
1.1	Integrability in $\mathcal{N} = 4$ Super Yang-Mills Theory	12
1.2	String model and integrability	14
1.3	Large volume spectrum from symmetry	16
1.4	Towards finite size	23
1.5	Bound States and Yangian	27
1.6	Different models	28
1.7	Outline	29
2	Integrable Models and Hopf Algebras	33
2.1	Classical Integrable Systems	33
2.2	Integrable 2d Relativistic Field Theories	35
2.3	Hopf Algebras	37
2.4	Yangians	40
2.5	Integrability and Yangians	42
3	Centrally extended $\mathfrak{su}(2 2)$	45
3.1	Defining Relations	45
3.2	Fundamental Representation	46
3.2.1	Matrix Realization	46
3.2.2	Parameterizations	47
3.3	The Outer Automorphism and $\mathfrak{gl}(2 2)$	48
3.4	Symmetric Short Representations	51
3.5	Hopf Algebra Structure	53
3.6	The Yangian of centrally extended $\mathfrak{su}(2 2)$	56
3.6.1	First realization	56

3.6.2	Second realization	58
3.7	Long Representations	61
	Appendix A: Exceptional Lie algebra	62
	Appendix B: Long Representation	65
4	Bound State S-Matrices	69
4.1	The Fundamental S-matrix	69
4.2	Kinematical Structure of the S-Matrix	72
4.2.1	Invariant subspaces	72
4.2.2	Basis and relations	74
4.3	Yangian Symmetry and Coproducts	77
4.3.1	(Opposite) coproduct basis	77
4.3.2	S-matrix in coproduct basis	79
4.4	Fundamental S-matrix revisited	80
4.5	Complete Solution of Case I	83
4.6	The S-matrix for Case II	87
4.7	Complete Solution of Case III	90
4.8	Reduction and Comparison	93
4.9	Summary	95
5	The Classical r-Matrix	97
5.1	The near plane-wave limit	98
5.2	The Moriyama-Torrielli proposal	99
5.3	The Beisert-Spill proposal	100
5.4	The semi-classical limit of the S-matrix	100
5.5	Comparison in the near plane-wave limit	104
5.6	Summary	105
6	Universal Blocks	107
6.1	The $\mathfrak{su}(2)$ subspace	107
6.2	Universal R-matrix for $\mathfrak{gl}(1 1)$	113
6.3	Summary	118
7	The Coordinate Bethe Ansatz	119
7.1	Formalism	119
7.1.1	Nonlinear Schrödinger Model	120
7.1.2	Adding color	123
7.2	The $\mathfrak{su}(2 2)$ (nested) coordinate Bethe Ansatz	128

7.2.1	Solving for the coefficients	129
7.3	Bethe Ansatz and Yangian Symmetry	132
7.3.1	Single excitations	132
7.3.2	Multiple excitations	134
7.3.3	Bethe equations	137
7.4	Summary	138
8	Algebraic Bethe Ansatz	139
8.1	Monodromy and transfer matrices	139
8.2	Diagonalization of the transfer matrix	141
8.2.1	Eigenvalue of the transfer matrix on the vacuum	141
8.2.2	Creation operators and excited states	144
8.2.3	Commutation relations	146
8.2.4	First excited state	148
8.2.5	General result and Bethe equations	150
8.3	Different vacua and fusion	152
8.3.1	Fermionic vacuum	153
8.3.2	Bosonic vacuum	154
	Appendix A: Algebraic Bethe ansatz for the 6-vertex model	159
	Appendix B: Excited states, $K^{\text{III}} = 0$	162

Chapter 1

Introduction

The current understanding of the microscopic world and gravity originates from the beginning of the twentieth century, when two revolutionary ideas saw the light of day. In 1900, Max Planck, with the quantum hypothesis, initiated a field that would become quantum mechanics and, in 1915, Albert Einstein formulated the theory of general relativity.

The theory of relativity replaced Newton's theory of gravity by considering space and time in a conceptually different way. Rather than having an ambient space in which masses feel gravity and move around, space and time are unified into an entity called space-time. Space-time is curved by its matter and energy content. Curvature is described by a metric which is an object that measures distances and angles. The metric is a dynamical quantity and it satisfies Einstein's equations of general relativity. General relativity successfully describes corrections to the orbits of planets that Newtonian gravity could not account for. It also predicts novel effects like gravitational lensing and gravitational time dilatation. These predictions were indeed confirmed by experiments.

Where general relativity was more or less a finished theory, it took quantum mechanics some years to mature into the theory that is part of the standard physics curriculum of universities today. Quantum mechanics radically changed the notions of particles and forces, since it describes nature in a probabilistic way. Outcomes of measurements can only be given in terms of probabilities and, moreover, measurements inevitably influence the system under study.

The quantum mechanical framework to describe systems with an infinite number of degrees of freedom is known as quantum field theory. Quantum field theories naturally include the concepts of particle production and annihilation. The large amount of degrees of freedom leads to superficial infinities that are caused by particles being created and annihilated at the same space-time point. To cope with these infinities, one employs the procedures of regularization and renormalization to obtain finite and measurable results for physical quantities.

By now, quantum field theory is one of the cornerstones of theoretical physics. Quantum field theories are used in the description of a wide variety of phenomena ranging from the interactions between fundamental particles to condensed matter systems. The parameters in a quantum field theory that describe the interaction strength between particles are called coupling constants. For example, in Quantum Electrodynamics the coupling constant between the electron and the photon is given by the charge of the electron e and it determines the strength of the electromagnetic force on the electron. Quantum field theories with small coupling constants are called weakly coupled. Weakly coupled quantum field theories are well-understood. They can be studied in a perturbative way, based on path integrals and Feynman diagrams. On the other hand, the knowledge of strongly coupled quantum field theories (i.e. large coupling constant) is only partial.

A relevant quantum field theory in which strongly coupled phenomena play a role is Quantum Chromodynamics (QCD). This theory describes the interactions between quarks and gluons. It is known to be asymptotically free, meaning that at very small distances quarks behave like free particles. In other words, in this regime, QCD is effectively a weakly coupled theory. However, at large distances, quarks couple strongly, which precludes perturbative methods to theoretically explain the phenomenon of confinement: why can hadrons - the bound states of quarks - be observed in nature, but not the free quarks? Other areas where strong interactions also are important, include for instance cold atomic gases and high T_c -superconductivity.

A class of quantum field theories that have remarkable properties are the supersymmetric quantum field theories. Supersymmetry is a symmetry between the bosons and fermions in a theory. Every boson in the theory has a fermionic partner and vice versa. The theory is invariant under the interchange of the particles and their superpartners.

A special supersymmetric theory, that plays an important role in this work, is $\mathcal{N} = 4$ super Yang-Mills theory ($\mathcal{N} = 4$ SYM). It is the maximally supersymmetric gauge theory in four space-time dimensions. The gauge group is $SU(N)$ and the theory has a single coupling constant g_{YM} . The rank N of the gauge group can be seen as a free positive integer-valued parameter. Introducing the 't Hooft coupling $\lambda = g_{YM}^2 N$, any perturbative Feynman diagrammatic expansion rearranges itself as [1]

$$Z = \sum_{n=0}^{\infty} N^{2-2n} \sum_{k=0} Z_{n,k} \lambda^k, \quad (1.1)$$

where the index n can be seen as the genus of the surface on which the corresponding diagram can be drawn. A particularly interesting limit to consider is $N \rightarrow \infty$ while keeping λ fixed. In this limit only the diagrams corresponding to zero genus contribute and for this reason it is called the planar limit. In the planar limit $\mathcal{N} = 4$ SYM exhibits the remarkable features of a solvable model.

What really sets $\mathcal{N} = 4$ SYM apart from a generic quantum field theory is that it exhibits conformal symmetry at the quantum level. This means in particular that the theory is invariant under rescalings. Conformal symmetry puts strong constraints on a theory. For example, it fixes two-point correlation functions of scalar fields to be of the form

$$\langle \mathcal{O}(x) \mathcal{O}(y) \rangle = \frac{1}{|x - y|^{2\Delta}}. \quad (1.2)$$

The constant Δ is called the scaling dimension and will depend non-trivially on the parameters λ, N . In general it admits a perturbative expansion

$$\Delta = \Delta_0 + \sum_{n=0} \sum_{m=1} \frac{\lambda^m}{N^{2n}} \Delta_{m,n}, \quad (1.3)$$

which in the planar limit becomes

$$\Delta = \Delta_0 + \sum_{m=1} \lambda^m \Delta_m. \quad (1.4)$$

Quantum field theories, in the form of the Standard Model, successfully describe the world of fundamental particles. How to construct a quantum theory of gravity, however, is currently unknown. One of the most viable candidates for a quantum theory of gravity is superstring theory.

The idea of string theory is to consider extended objects, called strings, rather than point-like particles as fundamental building blocks. A string can have the topology of a rod (open string) or of a ring (closed string). Different vibrations of a string describe different types of particles. One particularly interesting massless particle is found in the closed string spectrum. It carries spin two and can be identified with a graviton, a quantum of the gravitational field. Thus, string theory automatically incorporates gravity via the quantum mechanical modes of closed strings. Open strings can end on other extended objects called D-branes [2]. Massless excitation modes of such open strings give rise to gauge fields. Hence, open strings are naturally linked to gauge theories. In this way string theory, containing open and closed strings, offers a unified framework to treat gravity and gauge theories.

Superstring theories describe a string moving in a ten-dimensional space called the target-space. A propagating string in the target-space sweeps out a two-dimensional surface which is called the world-sheet. It can be parameterized by two parameters σ, τ . The σ -variable is the coordinate parameterizing the spatial extension along the string whereas τ corresponds to the time direction, see figure 1.1. The vibrations of superstrings can have bosonic and fermionic degrees of freedom, related to each other by supersymmetry transformations. Such string theories are said to have world-sheet supersymmetry. In addition, one can consider a string which propagates in a target-space that is a superspace (it has both bosonic and fermionic coordinates). The supersymmetric structure originating in this way is called target-space supersymmetry.

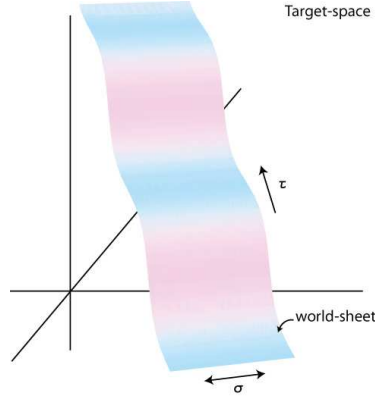


Figure 1.1: The world-sheet of an open string.

There are two coupling constants in string theory, called the string tension g and string coupling g_s . The string tension g describes the energy per unit length of the string and the string coupling g_s governs the splitting and joining processes of strings. When $g_s = 0$ there is no string splitting/joining and the world sheet of a closed string is just a cylinder.

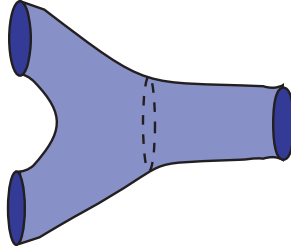


Figure 1.2: Schematic representation of a joining/splitting process of closed strings. Any such process is weighted with the coupling constant g_s .

An fascinating new insight in the dynamics of strings and strongly coupled gauge theories came with the advent of the AdS/CFT correspondence [3]. The correspondence states a duality between superstring theories of *closed strings* and conformal field theories. It assumes that a string theory in an anti-de Sitter (AdS) target-space is equivalent to a conformal field theory (CFT) on the conformal boundary of this space. What is remarkable about this conjecture is that it relates closed strings, which are inherently related to gravity, to a quantum field theory that has no gravitational degrees of freedom at all. In this way it provides a realization of a profound duality between open and closed strings.

We will now continue with describing the conjecture in more detail by considering the prototype example of the AdS/CFT correspondence:

$$\mathcal{N} = 4 \text{ super Yang-Mills} \Leftrightarrow \text{Type IIB AdS}_5 \times \text{S}^5 \text{ superstring.}$$

The $\text{AdS}_5 \times \text{S}^5$ superstring is the string theory in a special curved target space, which is the product of the five-dimensional anti-de Sitter space AdS_5 and the five-dimensional sphere S^5 , see figure 1.3. The AdS_5 space is a maximally symmetric, five-dimensional space with negative constant curvature. It can be viewed as a hyperboloid embedded in flat space, described by the equation $X_0^2 + X_5^2 - \sum_{i=1}^4 X_i^2 = 1$.

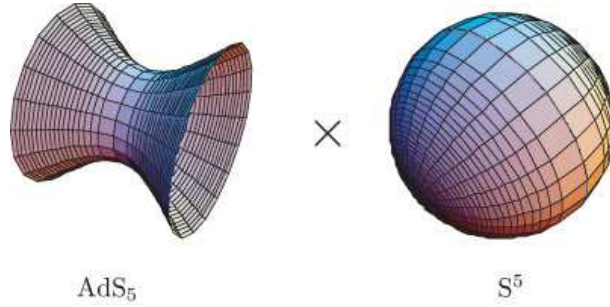


Figure 1.3: The $\text{AdS}_5 \times \text{S}^5$ space.

According to the AdS/CFT correspondence, the string coupling constant g_s and the string tension g are related to the $\mathcal{N} = 4$ SYM parameters, λ, N as

$$g_s = \lambda/4\pi N, \quad g = \sqrt{\lambda}/2\pi. \quad (1.5)$$

The correspondence also relates gauge invariant operators of $\mathcal{N} = 4$ SYM to string states. The scaling dimensions Δ of these operators are mapped to the energies E of corresponding string states

$$\Delta = E.$$

In other words, the spectrum of string energies should be equal to the spectrum of scaling dimensions of $\mathcal{N} = 4$ SYM. The problem of computing both spectra is naturally called the *spectral problem*.

The AdS/CFT correspondence is a strong-weak duality. This means that it relates the strongly coupled regime of field theories to the weakly coupled regime of the corresponding string theory and vice versa. As such, it enables to probe the strongly coupled regime of field theories via weakly-interacting strings, giving access to important strongly coupled phenomena in gauge theories. This potentially provides a direct way for string theory to have concrete applications, albeit more as a computational tool rather than as a fundamental model.

The strong-weak duality, however, is a two-edged sword. It offers truly exciting possibilities, yet it also presents a challenge in understanding the precise nature of the relation between strings and gauge theories. The reason is obvious; if one wishes to perform computations on both sides of the duality and compare the results, then generically one of the two sides will be strongly coupled and hence hard to solve. However, it turns out that both $\mathcal{N} = 4$ SYM and the $\text{AdS}_5 \times \text{S}^5$ superstring exhibit a rich hidden symmetry structure that allows to circumvent this problem. Namely, in the planar limit both $\mathcal{N} = 4$ super Yang-Mills and the $\text{AdS}_5 \times \text{S}^5$ superstring are described by integrable models. Integrable models are dynamical systems that have an infinite number of conservation laws, which normally imply the existence of an *exact* solution. One can therefore try to expand and generalize the methods and tools developed in the theory of integrable models, like for instance the Bethe ansatz, to explore and understand this prototype example at the quantitative level.

Thus, through the gauge/string correspondence, integrability offers the unprecedented possibility to completely solve, at least in the planar limit, a non-trivial quantum field theory in four dimensions. Understanding this prototype example would elucidate underlying physical mechanisms and would provide a solid basis to move on to other, more interesting, phenomenological models.

Integrability and its implementation for the $\text{AdS}_5 \times \text{S}^5$ superstring at the quantum level is by no means straightforward. The world-sheet theory turns out to be a two-dimensional non-relativistic quantum field theory in finite volume. Standard techniques appear to be insufficient for solving this theory and therefore new methods have to be invented. One of the most recent developments in this direction is the generalization of the so-called Thermodynamic Bethe Ansatz for the $\text{AdS}_5 \times \text{S}^5$ mirror model.

In the remainder of this chapter we will first discuss the emergence of integrability in both $\mathcal{N} = 4$ SYM and the $\text{AdS}_5 \times \text{S}^5$ superstring. Subsequently we will elaborate on how one can use integrability techniques to find a complete solution of the spectral problem. We close this presentation by giving an outline of this review and the new results achieved.

1.1 Integrability in $\mathcal{N} = 4$ Super Yang-Mills Theory

The constituents of the $\mathcal{N} = 4$ SYM theory are: six scalar fields Φ_i , one vector field A_μ and four fermions Ψ . The action is given by:

$$S = \frac{1}{g_{YM}^2} \int d^4x \left\{ \frac{1}{4} (F_{\mu\nu})^2 + \frac{1}{2} (D_\mu \Phi_i)^2 - \frac{1}{4} [\Phi_i, \Phi_j]^2 + \text{fermions} \right\}. \quad (1.6)$$

The fields are in the adjoint representation of $SU(N)$, thus under a local transformation $U(x) \in SU(N)$ they transform as

$$\mathcal{X} \rightarrow U\mathcal{X}U^{-1}, \quad A_\mu \rightarrow UA_\mu U^{-1} - i(\partial_\mu U)U^{-1}, \quad (1.7)$$

where $\mathcal{X} = \{\Phi_i, \Psi, F_{\mu\nu}\}$. We are interested in gauge invariant composite operators. They are formed by taking the trace over products of various fields \mathcal{X} , for instance

$$\mathcal{O}(x) = \text{tr}(\dots \Phi_i(x) \dots \Psi^k(x) D_\mu \Phi_j(x) \dots F_{\mu\nu}(x) \dots). \quad (1.8)$$

The symmetry algebra of $\mathcal{N} = 4$ SYM includes the conformal algebra $\mathfrak{so}(2, 4) \sim \mathfrak{su}(2, 2)$ which consists of the Poincaré algebra together with conformal boosts and dilatation. Adding the supersymmetry transformations extends the conformal algebra to the superconformal algebra $\mathfrak{psu}(2, 2|4)$. This is the full symmetry algebra of $\mathcal{N} = 4$ SYM.

A conformal field theory is characterized by the set of primary operators $\{\mathcal{O}_i\}$. These operators correspond to highest weight states, i.e. they are annihilated by conformal boosts and conformal supercharges of the superconformal algebra. Primary operators are eigenstates of the dilatation operator \mathcal{D} , which is one of the generators of the conformal algebra $\mathfrak{psu}(2, 2|4)$

$$\mathcal{D}\mathcal{O}_n = i\Delta_n\mathcal{O}_n. \quad (1.9)$$

In case these operators are scalar fields, one finds that their two-point function is of the form

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(y) \rangle = \frac{\delta_{ij}}{|x - y|^{2\Delta_i}}. \quad (1.10)$$

Thus, the spectrum of scaling dimensions can be determined either by finding eigenvalues of the dilatation operator or by computing the corresponding two-point functions.

Concerning the computation of two-point functions, one finds that generically the operators \mathcal{O}_i lose their tree-level orthogonality as soon as the leading quantum correction is taken into account. This can be understood as the appearance of a non-trivial mixing

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(y) \rangle = \frac{1}{|x - y|^{2\Delta_{\text{classical}}}} (\delta_{ij} + \lambda M_{ij} \ln(|\mu(x - y)|) + \dots), \quad (1.11)$$

where μ is a mass scale and M_{ij} is called mixing matrix. The spectrum of conformal dimensions is then found by re-diagonalizing the basis of operators. This procedure effectively introduces a dependence of the scaling dimension on the 't Hooft coupling $\Delta = \Delta(\lambda, N)$.

In the planar limit, a remarkable simplification happens. The dilatation operator can be identified with a spin chain Hamiltonian, while composite gauge invariant operators are then naturally interpreted as states of this spin chain [4–8].

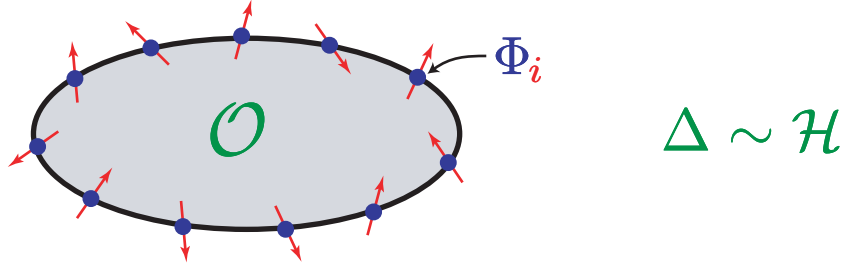


Figure 1.4: Operators correspond to states on a spin chain. The fields correspond to lattice sites. The scaling dimensions Δ are identified with eigenvalues of a Hamiltonian \mathcal{H}

To exemplify this, let us restrict to scalar fields at the one-loop level. One can make the identification

$$\text{tr}(\Phi_{i_1}(x) \dots \Phi_{i_J}(x)) \longrightarrow |\Phi_1\rangle \otimes \dots \otimes |\Phi_J\rangle, \quad (1.12)$$

which is a state of a $\mathfrak{so}(6)$ spin chain (the indices of the scalar fields transform under this algebra) with J sites. Cyclicity of the trace is then equivalent to the spin chain being closed.

By explicitly computing one-loop diagrams, one can show that the dilatation operator acts on neighboring fields only; it is of the form

$$\mathcal{D}_{1\text{-loop}} = \sum_{i=1}^J H_{i,i+1}. \quad (1.13)$$

This dilatation operator can be recognized as an integrable Hamiltonian of the $\mathfrak{so}(6)$ spin chain. Finding scaling dimensions thus reduces to computing the eigenvalues of this integrable spin chain Hamiltonian (figure 1.4). From the gauge theory point of view, the ground state of the Hamiltonian corresponds to

$$\text{tr}(\mathcal{Z}^J), \quad \mathcal{Z} \equiv \Phi_m + i\Phi_n, \quad (1.14)$$

for some $n, m = 1, \dots, 6$. This operator is known as half-BPS, which means that it is annihilated by half of the Poincaré supercharges. As a consequence of the superconformal algebra, its scaling dimension is $\Delta = J$ and it is not affected by quantum corrections.

1.2 String model and integrability

The action of the $\text{AdS}_5 \times \text{S}^5$ superstring string is of the form [9]

$$S = -\frac{g}{2} \int d\tau d\sigma g_{\mu\nu}(x) \partial^\alpha x^\mu \partial_\alpha x^\nu + \text{fermions}, \quad \alpha = \sigma, \tau$$

where g_{ij} is the metric of the $\text{AdS}_5 \times \text{S}^5$ space. Let (t, z^i) be the coordinates of AdS_5 and (ϕ, y^i) the coordinates of S^5 , then it is given by

$$ds^2 = g_{tt}dt^2 + g_{\phi\phi}d\phi^2 + g_{yy}dy^i dy^i + g_{zz}dz^i dz^i, \quad (1.15)$$

with

$$g_{tt} = \left(\frac{4 + z^2}{4 - z^2} \right)^2, \quad g_{\phi\phi} = \left(\frac{4 - y^2}{4 + y^2} \right)^2, \quad g_{yy} = \frac{1}{(1 + \frac{y^2}{4})^2}, \quad g_{zz} = \frac{1}{(1 - \frac{z^2}{4})^2}, \quad (1.16)$$

where $y^2 = y^i y^i$ and $z^2 = z^i z^i$. It is easily seen that the metric g_{ij} has isometries corresponding to shifts along the time direction t of the AdS space and to shifts along the big circle ϕ of the sphere. These correspond to global symmetries of the string model. The associated Noether charges are the energy E of the string and its angular momentum J respectively.

It is useful to note that we can write both AdS_5 and S^5 as cosets of Lie groups:

$$\text{AdS}_5 = \frac{\text{SO}(4, 2)}{\text{SO}(4, 1)}, \quad \text{S}^5 = \frac{\text{SO}(6)}{\text{SO}(5)}. \quad (1.17)$$

More precisely, the supergroup $\text{PSU}(2, 2|4)$ contains $\text{SU}(2, 2) \times \text{SU}(4)$ as a bosonic subgroup which is locally isomorphic to $\text{SO}(4, 2) \times \text{SO}(6)$. Modding $\text{PSU}(2, 2|4)$ out by $\text{SO}(4, 1) \times \text{SO}(5)$ then gives a supersymmetric space with $\text{AdS}_5 \times \text{S}^5$ as bosonic part. This is then a natural target-space for the $\text{AdS}_5 \times \text{S}^5$ superstring and indeed, the superstring on $\text{AdS}_5 \times \text{S}^5$ can be described on the coset [9]

$$\frac{\text{PSU}(2, 2|4)}{\text{SO}(4, 1) \times \text{SO}(5)},$$

see also [10] for an extensive review. One advantage of this formulation is that it makes the global $\mathfrak{psu}(2, 2|4)$ symmetry manifest. Notice that $\mathfrak{psu}(2, 2|4)$ is also the symmetry algebra of $\mathcal{N} = 4$ SYM.

One can prove that, classically, the string equations of motion admit a so-called Lax representation [11]. This property implies the existence of an infinite number of conservation laws. In other words, the $\text{AdS}_5 \times \text{S}^5$ superstring is classically integrable. The Lax representation allows one to explicitly construct the corresponding conserved charges [12] and find the solutions to the string equations of motion [13].

However, one ultimately is interested in the determination of the full quantum spectrum. At the quantum level, the $1 + 1$ dimensional quantum field theory on the world-sheet also shows signs of being integrable. The spectrum of spinning strings is compatible with the assumption of integrability [14–30] and scattering data of world-sheet excitations also seems to exhibit the properties of integrable theories [31–33]. From now on we will assume full quantum integrability of the model and try to understand its consequences. A necessary check of this assumption is

that all data that has been derived assuming integrability is in complete agreement with explicit computations.

To find the spectrum of quantum integrable field theories, one can employ the S-matrix approach, which proved to work well for many known integrable models. In the context of the AdS/CFT correspondence this approach was initiated in [34]. The AdS/CFT S-matrix and its symmetries constitute one of the main topics of this review. Below we will discuss a possible route one can undertake to find the string spectrum with the help of the S-matrix approach.

1.3 Large volume spectrum from symmetry

Integrability allows one to find the complete solution of the ‘large volume’ spectrum. On the gauge theory side this spectrum describes the scaling dimensions of operators composed of a large number of fields. On the string theory side, this gives the energy spectrum when the spatial size of the world-sheet goes to infinity. The S-matrix will play a crucial role in the derivation of the spectrum. How one obtains the spectrum is depicted in figure 1.5 and in what follows we will elaborate on the different steps.

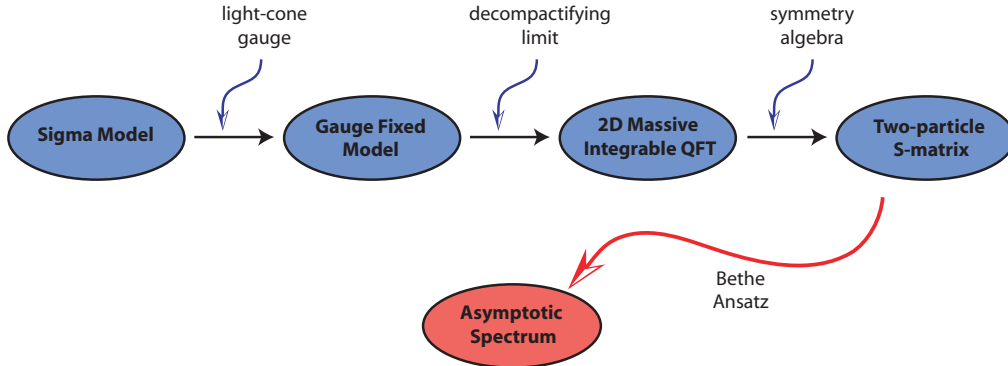


Figure 1.5: Road map for the asymptotic spectrum. For the gauge fixed model one finds in the infinite volume limit a massive integrable field theory. This theory has centrally extended $\mathfrak{su}(2|2)$ as symmetry algebra. The S-matrix can be found by requiring compatibility with this algebra and is then used to find the spectrum via the Bethe ansatz.

Symmetries

The energy of the string corresponds to the Noether charge associated with the time direction in the AdS space. It turns out that it can be related to a world-sheet Hamiltonian. This reduces the problem of finding the string energy to solving a spectral problem in a two-dimensional quantum field theory.

To see how this comes about, one works in the Hamiltonian formalism. We introduce the conjugate momenta p_i to a variable x^i

$$p_i = \frac{\delta S}{\delta(\partial_\tau x^i)}. \quad (1.18)$$

Consider the time direction t in the anti-de Sitter space and the angle ϕ describing one of the big circles of S^5 . The associated conserved charges E and J can be written in terms of the conjugate momenta

$$E \sim \int_{-r}^r d\sigma p_t, \quad J \sim \int_{-r}^r d\sigma p_\phi, \quad (1.19)$$

where we take the string world-sheet to be of size $-r \leq \sigma \leq r$.

To remove the non-physical degrees of freedom one imposes the so-called the light-cone gauge [35–37]. To this end, we define the light-cone coordinates

$$x_- = \phi - t, \quad x_+ = t, \quad p_- = p_\phi + p_t, \quad p_+ = p_\phi, \quad (1.20)$$

In these coordinates, the conserved charges can be expressed as follows

$$P_- \sim \int_{-r}^r d\sigma p_- = J - E, \quad P_+ \sim \int_{-r}^r d\sigma p_+ = J. \quad (1.21)$$

The light-cone gauge is now imposed by setting

$$x_+ = \tau, \quad p_+ = 1, \quad (1.22)$$

from which it follows that $r \sim P_+ = J$. In other words, the size of the world-sheet is proportional to the angular momentum of the string. To find the complete gauge fixed action, one solves the Virasoro constraints which give the light-cone momentum in terms of the transverse coordinates $p_-(x^i, x'^i)$. The world-sheet Hamiltonian density is given by

$$\mathcal{H} = -p_-(x^i, x'^i). \quad (1.23)$$

The world-sheet Hamiltonian is then related to the string energy and angular momentum via

$$H = \int_{-r}^r d\sigma \mathcal{H} = E - J. \quad (1.24)$$

Here one sees that the space-time energy E of the string is directly related to the spectrum of the world-sheet Hamiltonian H in this gauge. This means that one can find the spectrum of the superstring (and hence the spectrum of scaling dimensions of the dual four-dimensional field theory) by solving the spectral problem of the two-dimensional world-sheet theory.

For closed strings, periodicity of the fields implies that the total world-sheet momentum p defined by

$$p \equiv - \int_{-r}^r d\sigma p_i \partial_\sigma x^i \quad (1.25)$$

has to vanish. This condition is referred to as the level-matching condition. States that satisfy this condition are called on-shell and states with non-vanishing momentum are called off-shell. The level-matching condition can not be solved explicitly for the fields, but it needs to be imposed on physical states in the theory. One then proceeds by studying the off-shell theory, keeping in mind that for physical states the level-matching needs to be imposed at the end.

The next step is to consider the limit $P_+ \rightarrow \infty$ while keeping the string tension g fixed. In this limit, the world-sheet theory becomes a massive field theory defined on a plane. Because of this, asymptotic states and the S-matrix are well-defined. The gauged superstring has eight bosonic fields and eight fermionic fields.

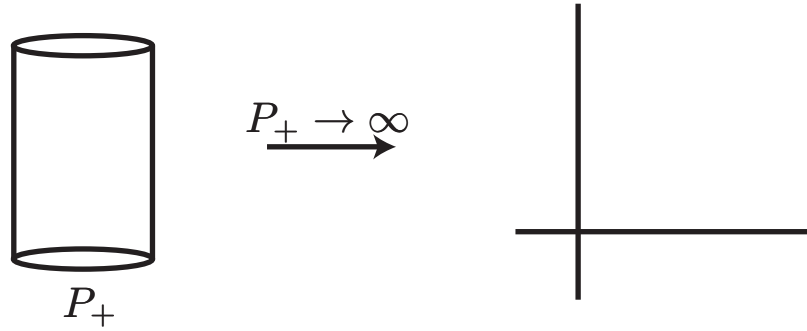


Figure 1.6: In the infinite volume limit theory is defined on a plane.

The gauge-fixing string model still has some residual symmetry left. It turns out that the $\mathfrak{psu}(2,2|4)$ algebra from the coset model is reduced to two copies of $\mathfrak{psu}(2|2)$ ¹ [39]. For the off-shell theory this symmetry algebra gets extended and becomes two copies of centrally extended $\mathfrak{su}(2|2)$. More precisely, the 16 degrees of freedom (8 bosons and 8 fermions)

$$8_B + 8_F = 16 = 4 \times 4. \quad (1.26)$$

transform under the tensor product of two fundamental representations of centrally extended $\mathfrak{su}(2|2)$. This fundamental representation is four dimensional and is spanned by two bosonic and two fermionic basis vectors.

The centrally extended $\mathfrak{su}(2|2)$ Lie superalgebra consists of two copies of $\mathfrak{su}(2)$, whose generators we denote by \mathbb{L}, \mathbb{R} together with two sets of supersymmetry generators $\mathbb{Q}, \mathbb{Q}^\dagger$. The

¹The same algebra also appears on the field theory side [38].

non-trivial commutation relations are given by

$$\begin{aligned}
[\mathbb{L}_a^b, \mathbb{J}_c] &= \delta_c^b \mathbb{J}_a - \frac{1}{2} \delta_a^b \mathbb{J}_c & [\mathbb{R}_\alpha^\beta, \mathbb{J}_\gamma] &= \delta_\gamma^\beta \mathbb{J}_\alpha - \frac{1}{2} \delta_\alpha^\beta \mathbb{J}_\gamma \\
[\mathbb{L}_a^b, \mathbb{J}^c] &= -\delta_a^c \mathbb{J}^b + \frac{1}{2} \delta_a^b \mathbb{J}^c & [\mathbb{R}_\alpha^\beta, \mathbb{J}^\gamma] &= -\delta_\alpha^\gamma \mathbb{J}^\beta + \frac{1}{2} \delta_\alpha^\beta \mathbb{J}^\gamma \\
\{\mathbb{Q}_\alpha^a, \mathbb{Q}_\beta^b\} &= \epsilon_{\alpha\beta} \epsilon^{ab} \mathbb{C} & \{\mathbb{Q}_a^{\dagger\alpha}, \mathbb{Q}_b^{\dagger\beta}\} &= \epsilon^{\alpha\beta} \epsilon_{ab} \mathbb{C}^\dagger \\
\{\mathbb{Q}_\alpha^a, \mathbb{Q}_b^{\dagger\beta}\} &= \delta_b^a \mathbb{R}_\alpha^\beta + \delta_\alpha^\beta \mathbb{L}_b^a + \frac{1}{2} \delta_b^a \delta_\alpha^\beta \mathbb{H},
\end{aligned} \tag{1.27}$$

where \mathbb{J} stands for any generator with appropriate index structure. The central extensions $\mathbb{C}, \mathbb{C}^\dagger$ make the anti-commutator between the supercharges non-zero. The central charges are related to the world-sheet momentum p and the string tension g as

$$\mathbb{C} = \frac{ig}{2}(e^{ip} - 1), \quad \mathbb{C}^\dagger = -\frac{ig}{2}(e^{-ip} - 1). \tag{1.28}$$

For on-shell configurations these central charges vanish.

Scattering

The existence of an infinite number of conservation laws greatly restricts scattering in an integrable theory. More precisely, scattering processes in integrable quantum field theories exhibit the following properties

- Absence of particle production and annihilation
- Conservation of the sets of initial and final momenta
- Factorization of multi-particle scattering into a sequence of two-body scattering events
- The two-body S-matrix satisfies a consistency condition which deals with equivalent orderings of scattering of multi-particle states called the Yang-Baxter equation.

These properties imply that, in integrable models, the two-body S-matrix is the fundamental building block of the scattering theory.

Having identified the symmetry properties of the gauge-fixed action, we should find its implications for scattering processes. The S-matrix relates *in*-eigenstates to *out*-eigenstates of the Hamiltonian and it should be compatible with the symmetry of the underlying model. This means that any such scattering matrix \mathbb{S} should commute with the action of any symmetry generator \mathbb{J}

$$\mathbb{S} \mathbb{J} = \mathbb{J} \mathbb{S}. \tag{1.29}$$

This is schematically depicted in figure 1.7. The action of the symmetry generators on the *in*- and *out*-states are encoded in a structure called the coproduct. The coproduct is an operation in

Hopf algebras which naturally encodes the action of symmetry generators on two-particle states. Since each of the world-sheet excitations transforms in a sixteen dimensional representation, the S-matrix will be a 256×256 dimensional matrix.

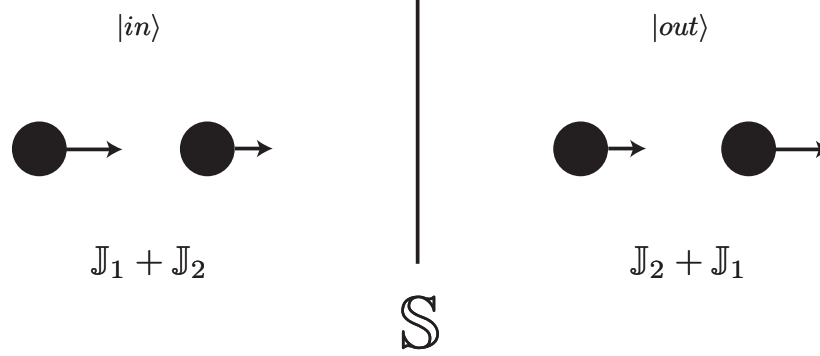


Figure 1.7: Symmetry commutes with scattering.

The remarkable fact is that requiring the two-body S-matrix for fundamental excitations to respect centrally extended $\mathfrak{su}(2|2)$ is enough to fix it up to an overall scalar factor S_0 [38, 40]. For its explicit form we refer to equation (4.1), but it can be shown that it satisfies all physical properties associated with integrable field theories

Unitarity: $S_{12}S_{21} = \mathbb{1}.$

Hermiticity: $S_{12}S_{12}^\dagger = \mathbb{1}.$

CPT Invariance: $S_{12} = S_{12}^t.$

Yang-Baxter Equation: $S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}.$

Based on the additional requirement of crossing symmetry [41], the overall scalar factor has also been conjectured [42–44] and was found to agree with all known computations so far.

Concluding, the expression for the full S-matrix is conjectured relying heavily on its symmetry properties. It is an exact quantity in the coupling g and hence interpolates between strong and weak coupling.

Large volume spectrum

In integrable models one usually can derive the exact large volume spectrum from the S-matrix by a technique called the Bethe Ansatz. This technique dates back to 1931, where it was first used to solve the Heisenberg XXX-spin chain [45]. Over the years this technique has been cast into many different forms and it has found its way into various physical models.

The Bethe Ansatz offers a method to capture the spectrum of an integrable Hamiltonian in a set of algebraic equations, called the Bethe ansatz equations. This is done by making a plane-wave type ansatz for the eigenstates of the Hamiltonian. This ansatz depends on a set of momenta which are restricted by imposing periodic boundary conditions. This is similar, for example, to a free particle on a circle of circumference L for which periodic boundary conditions $e^{ipL} = 1$ imply that the momentum is quantized. The reason periodicity needs to be imposed is that even though we consider the theory on the plane, it still comes from a closed string, i.e. a cylindric world-sheet. How to impose periodicity has a very clear interpretation.

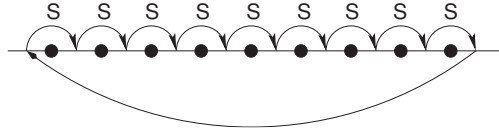


Figure 1.8: Periodicity is imposed by scattering a particle along the line.

Consider N particles with momenta p_i on a line with length L . If we take one particle and move it along the line, it will scatter on its way with all the other particles via the two-body S-matrix. However, when it has travelled a distance L , it returns back at its original position and the state should remain unchanged, up to a phase factor e^{ip_1L} , which is due to the plane-wave type ansatz. If we denote the S-matrix with S , then the Bethe equations are of the form

$$e^{ip_jL} = \prod_{i \neq j} S(p_i, p_j). \quad (1.30)$$

This can be seen as a quantization condition on the momenta. The spectrum is then obtained by first solving the Bethe equations for the set of momenta $\{p_i\}$, which can then be substituted in the Hamiltonian. The dispersion relation for the $\text{AdS}_5 \times \text{S}^5$ superstring is known in terms of the momenta p and is given by [46]

$$H = \sqrt{1 + 4g^2 \sin^2 \frac{p}{2}}. \quad (1.31)$$

The total energy of the state is obtained by summing the contributions of the particles

$$H_{tot} = \sum_i H(p_i). \quad (1.32)$$

The above discussion depends crucially on the two-particle S-matrix of the theory. Since this matrix is known for the $\text{AdS}_5 \times \text{S}^5$ superstring (in the large volume limit) to all orders in the coupling g , the spectrum obtained in this way is also exact.

The situation for the $\text{AdS}_5 \times \text{S}^5$ superstring appears to be more complicated than presented above. The S-matrix has a non-trivial matrix structure which mixes particles of different types. This can be taken into account by extending the Bethe ansatz resulting in a so-called nested

structure. The Bethe equations were found in [38, 47] and further investigated in [48, 49]. They are given by

$$\begin{aligned}
e^{ip_k L} &= \prod_{l=1, l \neq k}^{K^I} \left[S_0(p_k, p_l) \frac{x_k^+ - x_l^-}{x_k^- - x_l^+} \sqrt{\frac{x_l^+ x_k^-}{x_l^- x_k^+}} \right]^2 \prod_{\alpha=1}^2 \prod_{l=1}^{K_{(\alpha)}^{II}} \frac{x_k^- - y_l^{(\alpha)}}{x_k^+ - y_l^{(\alpha)}} \sqrt{\frac{x_k^+}{x_k^-}} \\
1 &= \prod_{l=1}^{K^I} \frac{y_k^{(\alpha)} - x_l^+}{y_k^{(\alpha)} - x_l^-} \sqrt{\frac{x_k^-}{x_k^+}} \prod_{l=1}^{K_{(\alpha)}^{III}} \frac{y_k^{(\alpha)} + \frac{1}{y_k^{(\alpha)}} - w_l^{(\alpha)} + \frac{i}{g}}{y_k^{(\alpha)} + \frac{1}{y_k^{(\alpha)}} - w_l^{(\alpha)} - \frac{i}{g}} \\
1 &= \prod_{l=1}^{K_{(\alpha)}^{II}} \frac{w_k^{(\alpha)} - y_k^{(\alpha)} - \frac{1}{y_k^{(\alpha)}} + \frac{i}{g}}{w_k^{(\alpha)} - y_k^{(\alpha)} - \frac{1}{y_k^{(\alpha)}} - \frac{i}{g}} \prod_{l \neq k}^{K_{(\alpha)}^{III}} \frac{w_k^{(\alpha)} - w_l^{(\alpha)} - \frac{2i}{g}}{w_k^{(\alpha)} - w_l^{(\alpha)} + \frac{2i}{g}},
\end{aligned} \tag{1.33}$$

where $\alpha = 1, 2$ reflect the two copies of $\mathfrak{su}(2|2)$ and $S_0(p_k, p_l)$ is the overall scalar factor of the S-matrix. The parameters x^\pm are related to the coupling g and to the world-sheet momentum via²

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i}{g}, \quad \frac{x^+}{x^-} = e^{ip}. \tag{1.34}$$

The parameters y_i, w_i are auxiliary variables that one introduces to deal with the matrix structure of the S-matrix. The spectrum is then again found by solving this coupled set of equations and plugging the solution in the dispersion relation (1.31).

Because one can only define the scattering theory on the infinite plane, these Bethe equations only encompass the asymptotic part of the spectrum. The big remaining challenge is then to compute the spectrum for finite size world-sheets.

Emergence of bound states

Before addressing the finite size problem, let us first take a closer look at the Bethe equations and discuss the emergence of bound states. Bound states are composite particles which belong to the physical spectrum and which manifest themselves as poles of a multi-particle S-matrix, see for example [50, 51]. It was found that the fundamental particles of the $\text{AdS}_5 \times \text{S}^5$ superstring model can indeed form bound states [52–56]. They transform in symmetric short representations of centrally extended $\mathfrak{su}(2|2)$ [55], which are discussed in chapter 3.

Consider two particles with complex momenta $p_1 = \frac{p}{2} - iq$ and $p_2 = \frac{p}{2} + iq$, with p real and $\text{Re } q > 0$. It is easy to see that in the large volume limit $e^{ip_1 J}$ tends to ∞ . When looking at the first line of the Bethe equations (1.33) we see that such a solution can indeed exist if the right hand side exhibits a pole

$$x^-(p_1) - x^+(p_2) = 0, \tag{1.35}$$

²A similar parameterization also appears in the Hubbard model.

which corresponds to a pole in the S-matrix. The above relation implies a non-trivial equation for p, q . From (1.35) one can show that the total energy becomes

$$H = H(p_1) + H(p_2) = \sqrt{2^2 + 4g^2 \sin^2 \frac{p}{2}}. \quad (1.36)$$

This discussion generalizes to multi-particle bound states. The latter are composites of ℓ fundamental particles whose momenta are related by the condition

$$x^-(p_i) - x^+(p_{i+1}) = 0. \quad (1.37)$$

The energy of these bound state particles is given by the dispersion relation

$$H = \sqrt{\ell^2 + 4g^2 \sin^2 \frac{p}{2}}. \quad (1.38)$$

A bound state that consists of ℓ fundamental particles and transforms in a short representation of centrally extended $\mathfrak{su}(2|2)$ which is 4ℓ dimensional. Concluding, the complete asymptotic spectrum consists of fundamental excitations and their bound states.

1.4 Towards finite size

To deal with the finite-size spectral problem, two approaches have been developed in the past in the context of relativistic models. One of them is a perturbative approach due to Lüscher [57] and the other is the Thermodynamic Bethe Ansatz (TBA) [58, 59]. Both approaches have been recently extended to account for the unconventional structure of the $\text{AdS}_5 \times \text{S}^5$ string model.

Lüscher's perturbative approach

In [57] the leading finite-size correction to energies were computed using diagrammatic methods. This formalism has been adapted to the $\text{AdS}_5 \times \text{S}^5$ superstring [60–65]. The idea is that, in compact spaces, particle energies pick up corrections coming from virtual particles moving around the compact direction, see figure 1.9. Where the particle meets the virtual particle from the loop they scatter via the S-matrix. The virtual particles that run in the loop can be both fundamental or bound state.

The first successful application of this procedure was the computation of the four-loop scaling dimension of the Konishi operator in $\mathcal{N} = 4$ SYM. The Konishi operator is of the form

$$K = \text{tr}(DZDZ) - \text{tr}(ZD^2Z), \quad (1.39)$$

where $D = D_1 + iD_2$. One can compute its scaling dimension directly in field theory [66–68]. It gives

$$\Delta_K = 4 + 3g^2 - 3g^4 + \frac{21}{4}g^6 + \left[-\frac{39}{4} + \frac{9\zeta(3)}{4} - \frac{45\zeta(5)}{8} \right] g^8. \quad (1.40)$$

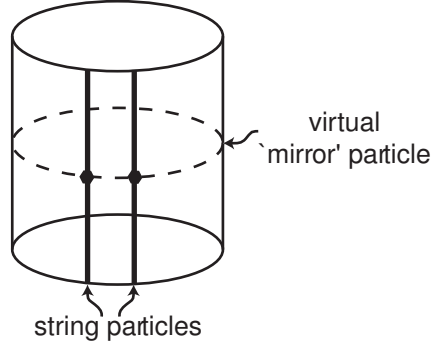


Figure 1.9: Diagrams for finite-size systems. The dashed line depicts a virtual particle moving around the compact direction.

To derive this result one has to take into account around 200 supergraphs or equivalently 130000 Feynman diagrams. Needless to say, these are complicated and demanding computations. However, this result can also be derived from string theory in a very simple and elegant way, as we will explain below.

First, one has to identify the string state to which the Konishi operator corresponds. On the string theory side, K corresponds to the state built up out of two world-sheet excitations corresponding to the light-cone derivatives D , with momenta p_1, p_2 . The level-matching condition implies that $p = p_1 = -p_2$. The angular momentum of the string is found to be $J = 2$. From the Bethe equations (1.33) one can solve perturbatively for the momentum p to find

$$p = \frac{2\pi}{3} - \frac{\sqrt{3}}{4}g^2 + \frac{9\sqrt{3}}{32}g^4 + \dots \quad (1.41)$$

From this momentum, one can then compute the energy via the dispersion relation (1.31)

$$E_{BAE} = J + H = 4 + \sqrt{1 + 4g^2 \sin^2 \frac{p_1}{2}} + \sqrt{1 + 4g^2 \sin^2 \frac{p_2}{2}}. \quad (1.42)$$

The field theory computation is done in perturbation theory at weak coupling, so in order to compare with this, one expands the energy E_{BAE} around $g = 0$ and finds

$$E_{BAE} = 4 + 3g^2 - 3g^4 + \frac{21}{4}g^6 + \left[-\frac{705}{64} + \frac{9\zeta(3)}{8} \right] g^8. \quad (1.43)$$

We see a disagreement in the g^8 term. However, it turns out that the Lüscher correction exactly contributes to this term. Indeed, when the Lüscher correction is taken into account, one *does* find perfect agreement [63]

$$E_{BAE+L} = 4 + 3g^2 - 3g^4 + \frac{21}{4}g^6 + \left[\frac{39}{4} + \frac{9\zeta(3)}{4} - \frac{45\zeta(5)}{8} \right] g^8. \quad (1.44)$$

Summarizing, by using the Bethe ansatz for the string model supplemented by the leading Lüscher correction, one finds beautiful agreement with the result of a highly non-trivial quantum field theory computation.

Although generating these nice results, Lüscher's approach is perturbative in nature and, therefore, has its limitations. The problem of establishing the exact spectrum is further addressed by the TBA approach, to which Lüscher's technique can be seen as a certain approximation. The regions of the (g, J) -parameter plane where the various techniques to study the string/gauge theory spectrum are applicable are schematically depicted in figure 1.10. The TBA should cover the entire diagram.

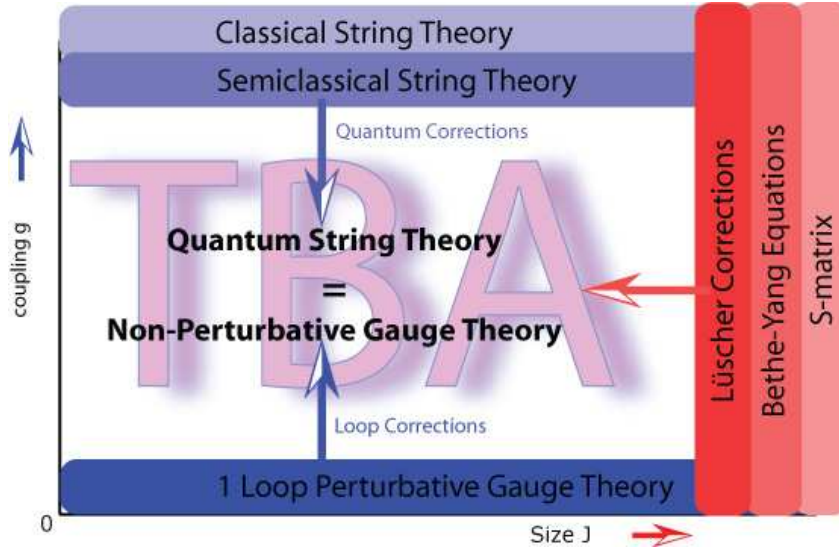


Figure 1.10: Overview of the known parts of the spectrum. The coupling constant g runs along the vertical axis and the size of the system on the horizontal. In the weak coupling regime perturbative gauge theory computations are possible and in the strong coupling regime one can perform perturbative string computations. For infinite sizes one can compute the asymptotic spectrum via Bethe ansatz techniques. The TBA should provide a complete covering of the diagram.

Thermodynamic Bethe Ansatz

The TBA approach in the AdS/CFT spectral problem was first advocated in [60] where it was used to explain wrapping effects in gauge theory. We will follow [56] where the first results towards an explicit construction of the corresponding TBA approach were obtained.

Consider the world-sheet of a closed string which is parameterized by variables σ and τ . In the thermodynamic Bethe ansatz (TBA) approach one considers a closed string of size L which

wraps a ‘time’-loop of size R . In this way one makes the time variable τ periodic. The surface that is formed in this way is a torus; the product of two circles of circumferences L, R , see figure 1.11.



Figure 1.11: String model and mirror model both come from the same torus. They are related via a double Wick rotation.

By performing a Wick rotation $\tau \rightarrow i\tau$ one obtains an Euclidean theory associated with this torus. In this Euclidean theory there is no distinction between the coordinates and because of this, one can associate two different ‘Minkowski’ models to this theory, namely one can apply an inverse Wick rotation to both variables.

Applying the inverse Wick rotation to the τ variable inevitably returns us back to the string model we started out with. Applying it to the σ variable, however, gives a new model, called ‘mirror’ model. This mirror model is then related to the original string model via a double Wick rotation

$$\tilde{\sigma} = -i\tau, \quad \tilde{\tau} = i\sigma. \quad (1.45)$$

Notice that the roles of position and time are interchanged. Consequently, the mirror model also has a different Hamiltonian \tilde{H} , which is now defined with respect to $\tilde{\tau}$.

It turns out that the mirror theory is different from the world-sheet theory of the $\text{AdS}_5 \times S^5$ superstring. For example, the dispersion relation is now given by

$$\tilde{H} = 2\text{arcsinh} \frac{\sqrt{1 + \tilde{p}^2}}{2g}, \quad (1.46)$$

where \tilde{p} is the momentum of the mirror particles, cf. (1.31). Nevertheless, one can show that the partition functions $Z(R, L), \tilde{Z}(L, R)$ of both models are equal. They are given by

$$Z(R, L) \equiv \sum_n \langle \psi_n | e^{-HR} | \psi_n \rangle = \sum_n e^{-E_n R} \quad (1.47)$$

$$\tilde{Z}(L, R) \equiv \sum_n \langle \tilde{\psi}_n | e^{-\tilde{H}L} | \tilde{\psi}_n \rangle. \quad (1.48)$$

In the limit $R \rightarrow \infty$ one obtains

$$\log Z(R, L) \xrightarrow{R \rightarrow \infty} -RE(L), \quad \log \tilde{Z}(R, L) \xrightarrow{R \rightarrow \infty} -RLf(L), \quad (1.49)$$

where $E(L)$ is the ground state energy and $f(L)$ is the free energy per unit length of the mirror model at temperature $1/L$. Since both partition functions agree, one finds

$$E(L) = Lf(L). \quad (1.50)$$

We see that by sending $R \rightarrow \infty$, the ground state energy of the $\text{AdS}_5 \times \text{S}^5$ superstring in finite volume L is described by the free energy of the mirror theory in infinite volume but at finite temperature $1/L$. This is the basic idea of the TBA; the spectrum of the original theory can be computed through thermodynamic quantities in the mirror model.

The importance of these observations is that, for the mirror model, one can still use all the large volume techniques that have been described earlier to compute the exact spectrum. In fact, the mirror model in the infinite volume limit also exhibits centrally extended $\mathfrak{su}(2|2)$ symmetry. This means that the same symmetry arguments are applicable to the large volume spectrum for the mirror model. Finally, one has to work at finite temperature and identify all states that contribute in the thermodynamic limit [69].

In the mirror theory, the S-matrix again contains poles that correspond to bound states [56]. In other words, the complete asymptotic spectrum is composed of fundamental particles and their bound states. More precisely, this means that bound states will contribute in the thermodynamic limit. Because of this, their scattering data and spectrum needs to be understood.

The thermodynamic limit is implemented by sending $R \rightarrow \infty$ while keeping N_i/R fixed, where N_i is the number of particles of a certain species i . From the Bethe equations that describe the large volume spectrum, one can then derive the TBA equations which constitute an infinite number of coupled integral equations. These equations are supposed to describe the finite-size spectrum of the original string theory [70–72]. The TBA approach perfectly accommodates Lüscher’s correction, which emerges from the large J asymptotic solutions of the TBA equations for excited states. The future challenge is to obtain a detailed understanding of the TBA solutions. The first results in this direction are very promising [73–79]. The agreement between the TBA approach and Lüscher’s approach has even been extended to the five-loop level [65, 80–82].

1.5 Bound States and Yangian

Bound states and their scattering data play an important role in both the TBA and in Lüscher’s perturbative approach. The S-matrix for fundamental representations was fixed by the require-

ment that it respects the $\mathfrak{su}(2|2)$ symmetry. One might wonder whether similar symmetry arguments can also be applied to bound states.

Bound states of fundamental particles transform in higher dimensional representations of centrally extended $\mathfrak{su}(2|2)$. Because of this, it turns out that invariance under the extended $\mathfrak{su}(2|2)$ algebra is no longer enough to fix the matrix structure for these higher dimensional representations [83, 84]. In addition, one has to invoke the Yang-Baxter equation to completely fix the S-matrix. However, it was shown that the fundamental S-matrix is actually invariant under a bigger symmetry group; the Yangian of $\mathfrak{su}(2|2)$ [83].

The Yangian of a Lie algebra can be seen as an infinite dimensional deformation of the associated loop algebra. Consider an algebra with structure constants f_C^{AB}

$$[\mathbb{J}^A, \mathbb{J}^B] = f_C^{AB} \mathbb{J}^C$$

and introduce a new set of generators $\hat{\mathbb{J}}^A$ that satisfy the relation

$$[\mathbb{J}^A, \hat{\mathbb{J}}^B] = f_C^{AB} \hat{\mathbb{J}}^C.$$

The Yangian of the algebra is now spanned by the generators $\mathbb{J}^A, \hat{\mathbb{J}}^A$ and commutators thereof, generating an infinite dimensional algebra. It turns out that bound state S-matrices are fixed by requiring invariance under the *Yangian* of centrally extended $\mathfrak{su}(2|2)$ rather than only the algebra itself [85]. In fact, one can construct any bound state S-matrix by using Yangian symmetry³ [88].

1.6 Different models

The main focus here is on the $\text{AdS}_5 \times S^5$ superstring and its partner $\mathcal{N} = 4$ super Yang-Mills. One might wonder how useful this is for physical applications since $\mathcal{N} = 4$ SYM is a rather special quantum field theory. It is conformal and highly supersymmetric, properties that are not shared with real-world theories like QCD. Nevertheless, one can smoothly deform the $\text{AdS}_5 \times S^5$ space-time [89] to obtain a deformed $\mathcal{N} = 1$ SYM theory. Even though the number of supersymmetries is reduced, one still finds integrable structures [90–92]. One might hope to apply the TBA approach to this case as well. There even exists a more general class of deformations [93] which is expected to be dual to a non-supersymmetric gauge theory. A thorough understanding of the prime example which is central in this work can then, via these related models, be used to extend our understanding of the AdS/CFT correspondence to more realistic models.

³Similarly, (Yangian) symmetry is also crucial in finding boundary S-matrices for open strings, see e.g. [86, 87].

A different example of a pair of dual theories is [94]

$$\mathcal{N} = 6 \text{ Chern-Simons Theory} \leftrightarrow \text{IIA strings on } \text{AdS}_4 \times \mathbb{CP}^3.$$

It also admits a formulation in terms of a coset model [95, 96] and one can show that the model is classically integrable. The asymptotic symmetry is again centrally extended $\mathfrak{su}(2|2)$ and, because of this, it has many features in common with the $\text{AdS}_5 \times S^5$ superstring and its partner $\mathcal{N} = 4$ SYM. Also for this model the asymptotic Bethe equations have been proposed [97] and the set of TBA equations have recently been conjectured [98, 99]. Even though there are quite some similarities between this instance of the AdS/CFT correspondence and the prototype example, there are also differences. Elucidating those can provide a partial guide to what new features could be expected on the way towards understanding more realistic models.

The importance of a complete understanding of the aforementioned theories goes beyond giving evidence for the AdS/CFT correspondence. On the one hand, the integrable structures can allow for a determination of the complete spectrum of $\mathcal{N} = 4$ SYM. This would then be the first exact solution of a non-trivial four-dimensional quantum field theory. On the other hand, a comprehensive description of the corresponding string model could prove invaluable in the future. A full solution might serve as a benchmark against which future computational methods in string theory can be checked and refined.

1.7 Outline

In this chapter we outlined a very promising road that hopefully leads to a full determination of the spectrum of the $\text{AdS}_5 \times S^5$ superstring and through the AdS/CFT correspondence to the spectrum of scaling dimensions of $\mathcal{N} = 4$ SYM. The material in this review particularly focusses on two aspects that were encountered along this way, namely scattering data of bound states and the Bethe ansatz.

First, the S-matrix describing the scattering of bound states with arbitrary bound state numbers is explicitly constructed by using Yangian symmetry, see equations (4.64), (4.88), (4.98). We study its classical limit and compare it against two different proposals in the literature [100, 101], finding agreement only with the latter. We also examine some of its mathematical features. More precisely we find blocks in this S-matrix that exhibit universal algebraic properties.

The next topic is the computation of the large volume spectrum of bound state configurations. The fact that we explicitly know the S-matrix allows us to apply the Bethe ansatz and impose periodic boundary conditions. This results in a set of Bethe equations that describe the large volume bound state spectrum (7.94). We have also found an alternative way to derive these equations by using the underlying Yangian symmetry of the S-matrix. These Bethe equations play a crucial role in the derivation of the TBA equations. A second approach that we have

examined is that of the algebraic Bethe ansatz. We present the explicit transfer matrices (8.56), from which one, once again, can derive the Bethe equations. The algebraic Bethe ansatz is also a useful framework for possible future investigations like form factors.

This remainder of this review is organized as follows. In chapter 2, we will first give a brief introduction to integrable models and Hopf algebras. In this chapter we also introduce the notion of a Yangian.

After this we will discuss in chapter 3 the algebra that plays a key role in the whole discussion: centrally extended $\mathfrak{su}(2|2)$. We will give its defining relations and discuss in detail the representations describing both fundamental particles and their bound states. Subsequently, we introduce the Hopf algebra structure that is used to derive the S-matrix. We finish the chapter with a discussion of the Yangian of $\mathfrak{su}(2|2)$, which is important for the determination of the bound state scattering data.

In chapter 4 the bound state S-matrix will be derived by making use of Yangian symmetry. Using the two $\mathfrak{su}(2)$ subalgebras contained in $\mathfrak{su}(2|2)$ one can split states into three different types. We first solve for one specific type of states (equation (4.64)) and then, via the different supersymmetry generators, extend the solution to the whole space, (equations (4.88) and (4.98)). This results in an explicit formula for the bound state scattering data that agrees with all S-matrices that were previously found.

Since the underlying algebra determines the entire scattering data, one might wonder whether it can be written purely in algebraic terms. Such an algebraic object corresponding to the S-matrix is called a universal R-matrix in the framework of Hopf algebras. This is the subject of chapters 5 and 6. In the first chapter the classical limit of the bound state S-matrix is studied and one finds agreement with a universal expression that has been proposed in the literature. In the subsequent chapter we study certain blocks in the S-matrix that exhibit universality at the full quantum level.

After this more mathematically oriented part we move on to the determination of the spectrum. We will first do this by applying the coordinate Bethe ansatz. After a discussion of this technique applied to the non-linear Schrödinger model we will introduce nesting to deal with particles with different colors and generalize the discussion to the $\text{AdS}_5 \times \text{S}^5$ superstring. This Bethe ansatz procedure can then be reformulated by making use of the Yangian symmetry, which allows for a derivation of the Bethe equations describing the asymptotic bound state spectrum [102].

Alternatively one can use the algebraic Bethe ansatz, which is done in chapter 8. In this approach one derives the eigenvalues of the transfer matrix, that play a crucial role in the TBA. We will derive an explicit expression for the eigenvalues of the transfer matrix for generic bound state representations (8.56). Some of these eigenvalues were already conjectured in the literature

[55] via a fusion procedure. We work this procedure out explicitly and compare it to our results, finding perfect agreement.

Integrable Models and Hopf Algebras

The notion of integrability was already briefly touched upon in the introduction. In this chapter we will expand this discussion. We will particularly focus on the relation between integrability and Hopf algebras. The mathematical language of Hopf algebras is useful in describing symmetries of integrable field theories and we will use it frequently.

We will first briefly discuss the notion of integrability in classical mechanics and in the theory of partial differential equations. We then continue with an overview of scattering processes in integrable field theories and introduce the notion of Hopf algebras and Yangians. After this we will show in an example how Yangians can arise in an integrable theory.

2.1 Classical Integrable Systems

Examples of classical integrable systems can be encountered *e.g.* when solving problems of Newtonian mechanics. Most of such problems, like Kepler's one, are well-known. However *exact* solutions, especially when dealing with multiple degrees of freedom, are rather rare. In the 19th century Liouville derived a theorem in which a big class of exactly solvable models was identified, the so-called integrable models. For an extensive treatment on this topic we refer to [103].

Finite-Dimensional Integrable Models

Consider a system with Hamiltonian \mathcal{H} , coordinates q_i and conjugate momenta p_i . Its equations of motion are written as

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}. \quad (2.1)$$

The time evolution of any quantity $F(p, q)$ is given by

$$\dot{F} = \{\mathcal{H}, F\}, \quad (2.2)$$

where $\{, \}$ is the Poisson bracket. Suppose that the phase space is $2N$ dimensional. The system is called *integrable* if there are N (functionally) independent conserved quantities F_i (of which the Hamiltonian is one), which mutually Poisson commute

$$\{F_i, F_j\} = 0, \quad \{F_i, \mathcal{H}\} = 0, \quad \text{for all } i, j. \quad (2.3)$$

Liouville's theorem states that an integrable system can be solved by quadratures, i.e. by solving a finite number of algebraic equations and computing a finite number of integrals. In this sense integrable models are exactly solvable.

Examples of integrable systems include the harmonic oscillator (the conserved quantity is the Hamiltonian $F = \mathcal{H}$) and Kepler's problem (conserved quantities are the Hamiltonian, the total angular momentum and the z-component of the angular momentum: $F_1 = \mathcal{H}, F_2 = J_3, F_3 = J^2$).

A different way to formulate integrability is in terms of a so-called Lax pair. Suppose there are matrices $L(p, q), M(p, q)$ such that one can write the equations of motion in the following way

$$\partial_0 L - [L, M] = 0. \quad (2.4)$$

(From now on we denote the time derivative as ∂_0 rather than using a dot). If this is the case, then one can straightforwardly see that the quantities

$$I_k = \text{tr} L^k \quad (2.5)$$

are conserved. The property that these quantities Poisson commute is related to an object called classical r -matrix. Assuming one can prove that these quantities are independent and Poisson commute with one another, we see that such a system is integrable.

The harmonic oscillator admits a Lax-pair. Define the matrices

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}, \quad M = \begin{pmatrix} 0 & -\frac{\omega}{2} \\ \frac{\omega}{2} & 0 \end{pmatrix}. \quad (2.6)$$

Then one can readily check that these indeed encode the equations of motion in the form (2.4). The Hamiltonian is given by $\mathcal{H} = \frac{1}{4} \text{tr} L^2$.

Since conserved charges correspond to symmetries via Noether's theorem, one finds that integrable models have enough symmetries to be solved by quadratures.

Integrable Partial Differential Equations

There is a related notion of integrability for two-dimensional (non-linear) partial differential equations (PDE). These equations admit a reformulation analogous to the Lax pair one discussed above.

Let $\Psi(t, x, z)$ be a rank n vector, and consider the overdetermined set of equations

$$\partial_0 \Psi(t, x, z) = L_0(t, x, z) \Psi(t, x, z), \quad \partial_1 \Psi(t, x, z) = L_1(t, x, z) \Psi(t, x, z), \quad (2.7)$$

where $\partial_1 = \partial_x$ and $L_i(t, x, z)$ are $n \times n$ matrices. By considering the double derivative $\partial_0 \partial_1 \Psi(t, x, z) = \partial_1 \partial_0 \Psi(t, x, z)$ one sees that the matrices $L_i(t, x, z)$ need to satisfy the consistency condition

$$\partial_0 L_1 - \partial_1 L_0 + [L_1, L_0] = 0. \quad (2.8)$$

The above equation can be seen as the flatness condition for a two-dimensional (non-abelian) connection. This connection is called the Lax connection and it is a generalization of the notion of a Lax pair (2.4).

If a PDE can be written as the flatness condition for a Lax connection, then it is called integrable. For example, the sine-Gordon equation

$$\phi_{tt} - \phi_{xx} + \frac{m^2}{\beta} \sin(\beta\phi) = 0, \quad (2.9)$$

can be written in this way via the following Lax connection:

$$L_1 = \frac{\beta\phi_t}{4i} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{k_0 \sin \frac{\beta\phi}{2}}{i} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{k_1 \cos \frac{\beta\phi}{2}}{i} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (2.10)$$

$$L_0 = \frac{\beta\phi_x}{4i} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{k_1 \sin \frac{\beta\phi}{2}}{i} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{k_0 \cos \frac{\beta\phi}{2}}{i} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (2.11)$$

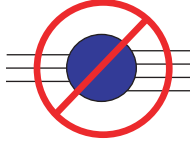
with

$$k_0 = \frac{m}{4} \left(z + \frac{1}{z} \right), \quad k_1 = \frac{m}{4} \left(z - \frac{1}{z} \right). \quad (2.12)$$

From a Lax connection one can construct an infinite tower of conserved charges. In this case this can be achieved by defining a monodromy matrix $T(z)$ as the path ordered exponential of the Lax connection. Expanding this quantity in the parameter z then generates the conserved quantities [103].

2.2 Integrable 2d Relativistic Field Theories

The notion of integrability can be extended to two dimensional quantum field theories. Also in these theories, integrability corresponds to the system having an enhanced symmetry resulting in an infinite set of conservation laws. This is translated into the fact that scattering processes in these theories have very special features. The scattering processes have the following properties [51, 104, 105]



Absence of Particle Production There is no particle production in these systems.

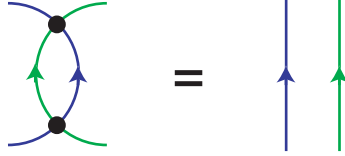
Momentum Conservation The sets of initial and final momenta are the same

$$\{p_i\}_{in} = \{p_i\}_{out}. \quad (2.13)$$

Factorizability Any scattering process reduces to a chain of two-body interactions

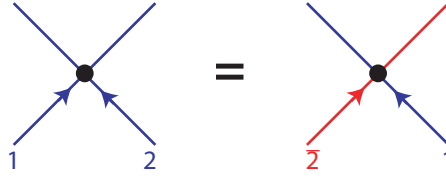
$$(n \rightarrow n) = \prod (2 \rightarrow 2). \quad (2.14)$$

Unitarity



$$\mathbb{S}_{12}(p_1, p_2) \mathbb{S}_{21}(p_2, p_1) = \mathbb{1}. \quad (2.15)$$

Crossing Scattering is symmetric under particle to anti-particle transformations



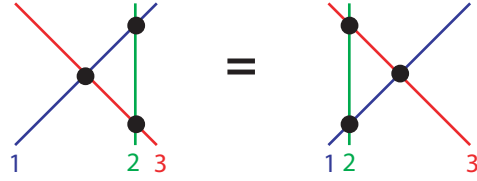
$$\mathbb{S}_{12}(p_1, p_2) = \mathbb{S}_{\bar{2}1}(p_1, \tilde{p}_2) \quad (2.16)$$

Yang-Baxter Equation The two body S-matrix satisfies a consistency condition which can be derived by considering the factorization of three particle scattering.

$$\mathbb{S}_{12} \mathbb{S}_{13} \mathbb{S}_{23} = \mathbb{S}_{23} \mathbb{S}_{13} \mathbb{S}_{12}. \quad (2.17)$$

One can often take the above set of conditions as a definition of an integrable (quantum) field theory. Examples of such a theories are for instance the Sine-Gordon model [106] and the Sinh-Gordon model [51].

From the factorization property we see that all scattering information in such theories is encoded in the two-body S-matrix. This means that computing this scattering process is the key to solving these systems.



2.3 Hopf Algebras

A convenient mathematical framework to deal with symmetries of integrable models is the language of Hopf algebras. Hopf algebras carry with them the structure of a coproduct and an antipode. From the physical point of view the coproduct describes the action of symmetry generators on multi-particle states and the antipode corresponds to a particle to anti-particle transformation.

We will review basic facts and definitions of Hopf algebras. We will be mostly interested in those associated to Lie (super)algebras. A very special family of Hopf algebras are the Yangians. These are infinite dimensional algebras associated to Lie algebras. They appear as symmetry algebras in various integrable models and are important in our understanding of the spectrum of the $\text{AdS}_5 \times \text{S}^5$ superstring.

There is a vast literature on this subject and for a more detailed analysis of Yangians and Hopf algebras we refer to [107–114].

Definitions

An associative algebra A with unit is a vector space over \mathbb{C} (the notion of algebra is more general, but will restrict to complex vector spaces) that is equipped with a multiplication μ

$$\mu : A \otimes A \rightarrow A, \quad a_1 \otimes a_2 \rightarrow \mu(a_1 \otimes a_2) \equiv a_1 a_2. \quad (2.18)$$

and a unit under multiplication $\eta : \mathbb{C} \rightarrow A$, such that

$$\mu(\eta(\lambda) \otimes a) = \lambda a = \mu(a \otimes \eta(\lambda)), \quad \lambda \in \mathbb{C}, a \in A. \quad (2.19)$$

The multiplication is bilinear and associativity is formulated as

$$a(bc) = (ab)c, \quad a, b, c \in A. \quad (2.20)$$

An obvious example of a such an algebra is the vector space of $n \times n$ complex matrices with the standard matrix multiplication and $\eta(\lambda) = \lambda \mathbb{1}$. Let $\mathcal{P} : A \otimes A \rightarrow A \otimes A$ be the (graded) permutation operator, then we say that the algebra is commutative if $\mu \circ \mathcal{P} = \mu$.

A coalgebra A is an object which has a structure ‘dual’ to that of an algebra; it has a comultiplication Δ and a co-unit ϵ

$$\Delta : A \rightarrow A \otimes A, \quad \epsilon : A \rightarrow \mathbb{C}. \quad (2.21)$$

Similarly these maps have to be bilinear and the coproduct needs to satisfy coassociativity

$$(\Delta \otimes \mathbb{1})\Delta = (\mathbb{1} \otimes \Delta)\Delta. \quad (2.22)$$

A coalgebra is called cocommutative if $\mathcal{P}\Delta = \Delta$. We call $\Delta^{op} = \mathcal{P}\Delta$ the opposite coproduct.

An algebra A is called a bialgebra if it also is endowed with the structure of a coalgebra such that Δ and ϵ are algebra homomorphisms, i.e. they respect the multiplicative structure

$$\Delta(a_1 a_2) = \Delta(a_1) \cdot \Delta(a_2), \quad \epsilon(a_1 a_2) = \epsilon(a_1)\epsilon(a_2), \quad (2.23)$$

$$\epsilon(\mathbb{1}) = 1, \quad \Delta \mathbb{1} = \mathbb{1} \otimes \mathbb{1}. \quad (2.24)$$

Finally a Hopf algebra H is a bialgebra which is equipped with an anti-homomorphism called the antipode $\mathcal{S} : A \rightarrow A$, which satisfies

$$\mathcal{S}(a_1 a_2) = \mathcal{S}(a_2)\mathcal{S}(a_1), \quad \mu(\mathcal{S} \otimes \mathbb{1}) \circ \Delta = \epsilon = \mu(\mathbb{1} \otimes \mathcal{S}) \circ \Delta, \quad (2.25)$$

where in the graded case one has to pick up relevant signs due to the graded structure. The first property is what defines \mathcal{S} to be an anti-homomorphism. An example of a Hopf algebra is the universal enveloping algebra of a Lie algebra. This algebra is automatically equipped with a multiplication, which in case of a matrix representation is just matrix multiplication. One can equip it with a Hopf algebra structure by specifying

$$\Delta(\mathbb{J}^A) = \mathbb{J}^A \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{J}^A, \quad \epsilon(\mathbb{J}^A) = 0, \quad \mathcal{S}(\mathbb{J}^A) = -\mathbb{J}^A, \quad (2.26)$$

and one can use the homomorphism properties to extend the above maps to products of elements. It is easy to check that the maps defined in this way respect multiplication (and hence also the Lie bracket). This Hopf algebra is cocommutative, but generically not commutative as an algebra.

The coproduct offers a natural way to extend a representation of a Lie algebra \mathfrak{g} on a vector space V to a tensor product representation on $V \otimes V$. One can use $(\Delta \otimes \mathbb{1})\Delta$ to define it on a triple tensor product and so on. By coassociativity one gets the same structure from using $(\mathbb{1} \otimes \Delta)\Delta$.

The above definition of the coproduct for the universal enveloping algebra (2.26) is natural in the language of symmetries in physics. For example, consider two particles $|m_1\rangle, |m_2\rangle$ in quantum mechanics with z-component of the angular momenta m_1, m_2 . One expects to find that in their tensor product state $|m_1\rangle \otimes |m_2\rangle$ their charges m_1 and m_2 add. The operator \hat{S}_z is just an element of $\mathfrak{su}(2)$, in other words one has

$$\hat{S}_z(|m_1\rangle \otimes |m_2\rangle) \equiv \hat{S}_z|m_1, m_2\rangle = (m_1 + m_2)|m_1, m_2\rangle = \Delta S_z|m_1, m_2\rangle, \quad (2.27)$$

by the above definition of the coproduct (2.26). This indeed indicates that the coproduct encodes the natural action of symmetry generators on multi-particle states.

Quasitriangular Hopf algebras

To any Hopf algebra $(H, \epsilon, \mathcal{S}, \Delta)$ one can associate another Hopf algebra $(H, \epsilon, \mathcal{S}, \Delta^{op})$ by equipping it with the opposite coproduct Δ^{op} rather than Δ . This Hopf algebra is called the opposite Hopf algebra H^{op} .

Generically there need not be a relation between the two different Hopf algebra structures. However, there is a class of Hopf algebras called quasi-cocommutative Hopf algebras, where there is an invertible element $\mathbb{S} \in H \otimes H$ such that

$$\Delta^{op} \mathbb{J} \mathbb{S} = \mathbb{S} \Delta \mathbb{J}, \quad \mathbb{J} \in H. \quad (2.28)$$

The element \mathbb{S} is called the R-matrix in mathematics. In physics this object corresponds to the S-matrix. The motivation for this is as follows. In physics the S-matrix relates in-states to out-states of the Hamiltonian.

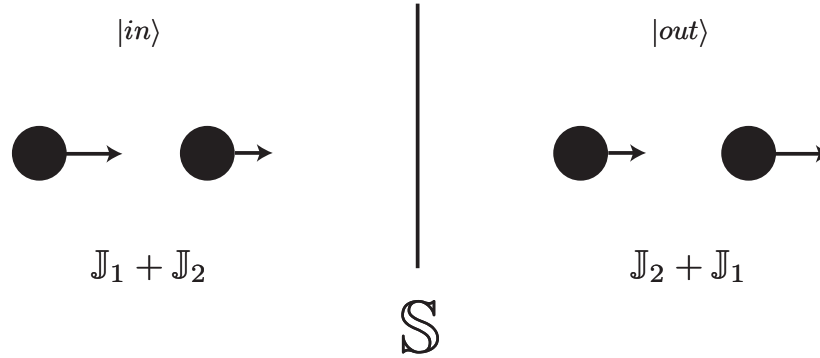


Figure 2.1: Symmetry commutes with scattering.

For definiteness, consider an elastic scattering process of two to two particles. The in-state can then be seen as two particles with the fastest one to the left and the slowest one on the right, figure 2.1. After they scatter, the particles have changed places and the fastest one is now on the right. If the theory possesses some symmetry algebra, then the S-matrix should be compatible with this algebra. This now translates into (2.28). In the context of integrable models, this can be made more precise by using the so-called Faddeev-Zamolodchikov algebra [104, 115].

Suppose now that the R-matrix satisfies the extra conditions

$$(\Delta \otimes \mathbb{1})(\mathbb{S}) = \mathbb{S}_{13} \mathbb{S}_{23}, \quad (\mathbb{1} \otimes \Delta)(\mathbb{S}) = \mathbb{S}_{13} \mathbb{S}_{12}, \quad (2.29)$$

then the Hopf algebra is called quasitriangular. The R-matrix of a quasitriangular Hopf algebra satisfies the following properties

$$\mathbb{S}_{12} \mathbb{S}_{13} \mathbb{S}_{23} = \mathbb{S}_{23} \mathbb{S}_{13} \mathbb{S}_{12}, \quad (\mathcal{S} \otimes \mathbb{1}) \mathbb{S} = (\mathbb{1} \otimes \mathcal{S}^{-1}) \mathbb{S} = \mathbb{S}^{-1}. \quad (2.30)$$

The first relation is easily recognized as the Yang-Baxter equation (2.17) and the second corresponds to the crossing equation (2.16). These equations were important properties of scattering processes in integrable field theories and they arise naturally in the context of Hopf algebras.

In the above we assumed that the R-matrix was an element of $H \otimes H$, but the abstract form of this element might be hard to find. However, in a specific representation, the intertwining R-matrix can usually be computed explicitly. In what follows we will refer to the R-matrix seen as an abstract element in $H \otimes H$ as the universal R-matrix. The intertwining operator in an explicit representation will be referred to as the S-matrix.

2.4 Yangians

A class of Hopf algebras that play an important role in integrable systems are the so-called Yangians. The Yangians are a family of infinite dimensional algebras that are associated to Lie algebras. They are constructed by introducing an additional set of generators to the Lie algebra ones. In this section we collect some basic facts about Yangians. For more details see e.g [109, 110, 113, 114].

Definition

Consider the universal enveloping algebra $U(\mathfrak{g})$ of a (simple) Lie algebra \mathfrak{g} , with structure constants f_C^{AB}

$$[\mathbb{J}^A, \mathbb{J}^B] = f_C^{AB} \mathbb{J}^C. \quad (2.31)$$

The Yangian $Y(\mathfrak{g})$ of this Lie algebra is the algebra generated by the generators \mathbb{J} of \mathfrak{g} and a new set of generators $\hat{\mathbb{J}}$, subject to

$$[\mathbb{J}^A, \hat{\mathbb{J}}^B] = f_C^{AB} \hat{\mathbb{J}}^C. \quad (2.32)$$

Higher level generators can then be obtained by commuting two level one generators and so on. The above commutation relations should obey the Jacobi and Serre relations (these are for algebras other than $\mathfrak{su}(2)$)

$$\begin{aligned} [\mathbb{J}^{[A}, [\mathbb{J}^B, \mathbb{J}^{C]}] &= 0 \\ [\mathbb{J}^{[A}, [\mathbb{J}^B, \hat{\mathbb{J}}^{C]}] &= 0 \\ [\hat{\mathbb{J}}^{[A}, [\hat{\mathbb{J}}^B, \mathbb{J}^{C]}] &= \frac{1}{4} f_D^{AG} f_E^{BH} f_F^{CK} f_{GHK} \mathbb{J}^{(D} \mathbb{J}^E \mathbb{J}^{F)}, \end{aligned}$$

where $()$, $[]$ in the indices stand for total symmetrization and anti-symmetrization, respectively. The indices of the structure constants are lowered with the Cartan-Killing matrix.

The Yangian can be given the structure of a Hopf algebra by specifying the following co-product

$$\Delta \mathbb{J}^A = \mathbb{J}^A \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{J}^A \quad \Delta \hat{\mathbb{J}}^A = \hat{\mathbb{J}}^A \otimes \mathbb{1} + \mathbb{1} \otimes \hat{\mathbb{J}}^A + \frac{1}{2} f_{BC}^A \mathbb{J}^B \otimes \mathbb{J}^C. \quad (2.33)$$

The coproducts of the higher level generators are obtained by using the fact that Δ should respect commutators. Then antipode is given by

$$\mathcal{S}(\mathbb{J}^A) = -\mathbb{J}^A \quad \mathcal{S}(\hat{\mathbb{J}}^A) = -\hat{\mathbb{J}}^A + \frac{1}{4} f_{BC}^A f_D^{BC} \mathbb{J}^D, \quad (2.34)$$

and the counit is

$$\epsilon(\mathbb{J}^A) = \epsilon(\hat{\mathbb{J}}^A) = 0 \quad \epsilon(\mathbb{1}) = 1. \quad (2.35)$$

This realization of the Yangian is called Drinfeld's first realization.

Drinfeld's second realization

There is also a second realization of a Yangian. This realization turns out to be particularly useful when checking the Serre relations of a representation explicitly or in constructing the universal R-matrix [116, 117].

The second realization of the Yangian [114] is given in terms of Chevalley-Serre type generators $\kappa_{i,m}, \xi_{i,m}^\pm$, $i = 1, \dots, \text{rank } \mathfrak{g}$, $m = 0, 1, 2, \dots$ satisfying relations

$$\begin{aligned} [\kappa_{i,m}, \kappa_{j,n}] &= 0, \quad [\kappa_{i,0}, \xi_{j,m}^+] = a_{ij} \xi_{j,m}^+, \\ [\kappa_{i,0}, \xi_{j,m}^-] &= -a_{ij} \xi_{j,m}^-, \quad [\xi_{j,m}^+, \xi_{j,n}^-] = \delta_{i,j} \kappa_{j,n+m}, \\ [\kappa_{i,m+1}, \xi_{j,n}^+] - [\kappa_{i,m}, \xi_{j,n+1}^+] &= \frac{1}{2} a_{ij} \{\kappa_{i,m}, \xi_{j,n}^+\}, \\ [\kappa_{i,m+1}, \xi_{j,n}^-] - [\kappa_{i,m}, \xi_{j,n+1}^-] &= -\frac{1}{2} a_{ij} \{\kappa_{i,m}, \xi_{j,n}^-\}, \\ [\xi_{i,m+1}^\pm, \xi_{j,n}^\pm] - [\xi_{i,m}^\pm, \xi_{j,n+1}^\pm] &= \pm \frac{1}{2} a_{ij} \{\xi_{i,m}^\pm, \xi_{j,n}^\pm\}, \\ i \neq j, \quad n_{ij} = 1 + |a_{ij}|, \quad \text{Sym}_{\{k\}}[\xi_{i,k_1}^\pm, [\xi_{i,k_2}^\pm, \dots [\xi_{i,k_{n_{ij}}}^\pm, \xi_{j,l}^\pm] \dots]] &= 0. \end{aligned} \quad (2.36)$$

In these formulas, a_{ij} is the (symmetric) Cartan matrix. The index m in the generators is referred to as the level.

Drinfeld [114] gave an explicit isomorphism between the two realizations as follows. Let us define a Chevalley-Serre basis for \mathfrak{g} as composed of Cartan generators \mathfrak{H}_i , and positive (negative) simple roots \mathfrak{E}_i^+ (\mathfrak{E}_i^- , respectively). Then one has

$$\begin{aligned} \kappa_{i,0} &= \mathfrak{H}_i, \quad \xi_{i,0}^+ = \mathfrak{E}_i^+, \quad \xi_{i,0}^- = \mathfrak{E}_i^-, \\ \kappa_{i,1} &= \hat{\mathfrak{H}}_i - v_i, \quad \xi_{i,1}^+ = \hat{\mathfrak{E}}_i^+ - w_i, \quad \xi_{i,1}^- = \hat{\mathfrak{E}}_i^- - z_i, \end{aligned} \quad (2.37)$$

where v_i, w_i, z_i are certain quadratic combinations of level-zero generators that we will not list here explicitly. From the level-zero and level-one generators, one can then recursively construct all higher-level generators by repeated use of the relations (2.36).

We will employ both the first and second realization of the Yangian in the other chapters.

Evaluation representation

An important representation of the Yangian $Y(\mathfrak{g})$ is the evaluation representation. Let us work in Drinfeld's first realization. Consider a representation V of \mathfrak{g} and introduce a parameter u . Then $V(u)$ can host a so-called evaluation representation of $Y(\mathfrak{g})$ by setting

$$\hat{\mathbb{J}}^A|v\rangle = u\mathbb{J}^A|v\rangle, \quad |v\rangle \in V. \quad (2.38)$$

Of course in order for this to be a valid representation, one needs to check that the Serre relations are satisfied.

Double Yangian

The Yangian $Y(\mathfrak{g})$ of a Lie (super)algebra is not quasitriangular, i.e. there is no element $\mathbb{S} \in Y(\mathfrak{g}) \otimes Y(\mathfrak{g})$ that intertwines the coproduct with the opposite coproduct. However, there exists a Hopf algebra called the double Yangian $DY(\mathfrak{g})$ which is defined by relations (2.36) but now one takes the level $m \in \mathbb{Z}$ [114, 116]. On evaluation representations this means that also negative powers of the evaluation parameter u are considered.

This enhanced algebra is quasitriangular for simple Lie algebras [116]. For the Yangian of Lie superalgebras this is not known, although it has been found in specific cases see for instance [108, 117–119]. The Yangian $Y(\mathfrak{g})$ can be identified with a subalgebra of $DY(\mathfrak{g})$ by restricting to elements with positive level. This means that in any representation, the universal R-matrix of $DY(\mathfrak{g})$ also gives the S-matrix that relates the coproduct and opposite coproduct for $Y(\mathfrak{g})$.

2.5 Integrability and Yangians

We have already seen that the R-matrix of a quasitriangular Hopf algebra is closely related to the S-matrix of integrable systems. Yangians also appear naturally in the context of integrable models. They appear in a variety of systems like the Hubbard model [120], the XXX spin chain [121] and also, as we will see later, in the $\text{AdS}_5 \times \text{S}^5$ superstring [83]. In this section we will treat an example of how a Yangian can arise in integrable models [109].

Consider a two-dimensional field theory with Noether currents taking values in some Lie-algebra \mathfrak{g}

$$J_\mu = J_{\mu;a}(t, x)t^a, \quad t \in \mathfrak{g}. \quad (2.39)$$

These currents are conserved on-shell

$$\partial^\mu J_\mu = 0, \quad (2.40)$$

and hence they define conserved charges

$$Q_a = \int dx J_{0;a}. \quad (2.41)$$

Assume that the currents satisfy the flatness condition

$$\partial_0 J_1 - \partial_1 J_0 + [J_0, J_1] = 0. \quad (2.42)$$

Both conservation of the current and the above condition (2.42) are equivalent to the flatness of the following Lax connection

$$L_\mu(t, x, z) = \frac{1}{1 - z^2} (J_\mu(t, x) + z \epsilon_\mu^\nu J_\nu(t, x)). \quad (2.43)$$

In this sense this model is integrable. An example of such a model is the principal chiral model see e.g. [10, 122]. Due to integrability one expects to find more conserved charges than just the ones corresponding to the Lie algebra. Because of the flatness condition it turns out that one can define an additional non-local current

$$\hat{J}_{\mu;a}(t, x) = \epsilon_{\mu\nu} J_a^\nu(t, x) - \frac{i}{2} f_{abc} J_{\mu;b}(t, x) \int_{-\infty}^x dy J_{0;c}(t, y). \quad (2.44)$$

One can then show that the corresponding charge

$$\hat{Q}_a = \int dx \hat{J}_{0;a}(t, x) \quad (2.45)$$

is conserved by using conservation of the current J_μ , integration by parts and finally the flatness condition.

The conserved charges \hat{Q}_a form an algebra which can be studied by computing their Poisson brackets. We will now study the Hopf algebra structure of this model. Assume that there are particle-like solutions of the equations of motion that can be taken to be well separated, see figure 2.2.

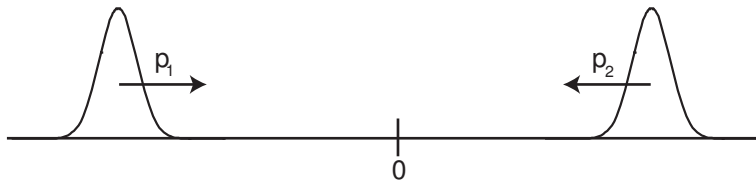


Figure 2.2: A schematic state of a pair of well-separated particles.

Let us first compute the charges forming the Lie algebra \mathfrak{g} on such a profile:

$$Q_a|_{\text{profile}} = \int_{-\infty}^0 dx J_{0;a}(t, x) + \int_0^{\infty} dx J_{0;a}(t, x). \quad (2.46)$$

This can be thought of as a semi-classical analog of the coproduct of Q , and we recognize

$$\begin{aligned} \Delta Q_a &= Q_a|_{\text{particle 1}} + Q_a|_{\text{particle 2}} \\ &= Q_a \otimes \mathbb{1} + \mathbb{1} \otimes Q_a. \end{aligned} \quad (2.47)$$

This exactly agrees with the coproduct for the enveloping algebra that was discussed earlier (2.26).

For the non-local charges \hat{Q}_a the discussion becomes more involved. When evaluating this charge on the same profile we can split the integral

$$\begin{aligned} \Delta \hat{Q}_a &= \int_{-\infty}^0 dx J_a^1(x) + \int_0^{\infty} dx J_a^1(x) - \frac{if_{abc}}{2} \left\{ \int_{-\infty}^0 dx J_{0;b}(x) \int_{-\infty}^x dy J_{0;c}(y) + \right. \\ &\quad \left. + \int_0^{\infty} dx J_{0;b}(x) \int_0^x dy J_{0;c}(y) + \int_0^{\infty} dx J_{0;b}(x) \int_{-\infty}^0 dy J_{0;c}(y) \right\}. \end{aligned} \quad (2.48)$$

The first terms clearly give $\hat{Q}_a|_{\text{particle 1}} + \hat{Q}_a|_{\text{particle 2}}$ and the last piece is given in terms of the charges Q_a . We can write this as

$$\Delta \hat{Q}_a = \hat{Q}_a \otimes \mathbb{1} + \mathbb{1} \otimes \hat{Q}_a - \frac{i}{2} f_{abc} Q_b \otimes Q_c. \quad (2.49)$$

Here one recognizes the coproduct of a Yangian generator (2.33).

Since integrability is closely related to symmetries, it is important to know the underlying symmetry algebra of an integrable model. For the $\text{AdS}_5 \times \text{S}^5$ superstring in the decompactifying limit this algebra consists of two copies of centrally-extended $\mathfrak{su}(2|2)$ [39]. Since the classical model allows a Lax reformulation [11] one expects, in view of the above example, Yangian symmetry to be present. This indeed turns out to be the case. In the next chapter we will study centrally-extended $\mathfrak{su}(2|2)$ and its Yangian in detail.

Chapter 3

Centrally extended $\mathfrak{su}(2|2)$

The algebra which plays a key role in the developments we will present is the centrally extended $\mathfrak{su}(2|2)$ Lie superalgebra, which we will denote with \mathfrak{h} throughout the rest of this work. This chapter will be devoted to the discussion of the basic properties of this algebra. The main focus will be on symmetric representations and the underlying Hopf algebra structure. \mathfrak{h} is the symmetry algebra of the light-cone Hamiltonian of the $\text{AdS}_5 \times \text{S}^5$ superstring [39] and it also appears as the symmetry algebra of the spin chain describing single-trace operators in $\mathcal{N} = 4$ super Yang-Mills [38].

We will first discuss the definition of this algebra and its automorphisms. After this we will describe its representations, their tensor products and the (twisted) Hopf algebra structure. The last part of this chapter will be dealing with the Yangian of \mathfrak{h} and its corresponding Hopf algebra structure. This mathematical framework forms the basis for subsequent chapters.

In this chapter we will amply utilize the language of Hopf algebras and Yangians, cf. chapter 2.

3.1 Defining Relations

\mathfrak{h} consists of two sets of bosonic generators $\mathbb{R}_\beta^\alpha, \mathbb{L}_b^a$ that constitute two copies of the $\mathfrak{su}(2)$ algebra. We will use the convention that roman letters denote bosonic indices and take values $a, b, \dots = 1, 2$ and greek letters are used for fermionic indices and take values $\alpha, \beta, \dots = 3, 4$. There are supersymmetry generators $\mathbb{Q}_\alpha^a, \mathbb{Q}_a^{\dagger\alpha}$ and central elements $\mathbb{H}, \mathbb{C}, \mathbb{C}^\dagger$. The non-trivial commutation

relations between the generators are given by

$$\begin{aligned}
[\mathbb{L}_a^b, \mathbb{J}_c] &= \delta_c^b \mathbb{J}_a - \frac{1}{2} \delta_a^b \mathbb{J}_c & [\mathbb{R}_\alpha^\beta, \mathbb{J}_\gamma] &= \delta_\gamma^\beta \mathbb{J}_\alpha - \frac{1}{2} \delta_\alpha^\beta \mathbb{J}_\gamma \\
[\mathbb{L}_a^b, \mathbb{J}^c] &= -\delta_a^c \mathbb{J}^b + \frac{1}{2} \delta_a^b \mathbb{J}^c & [\mathbb{R}_\alpha^\beta, \mathbb{J}^\gamma] &= -\delta_\alpha^\gamma \mathbb{J}^\beta + \frac{1}{2} \delta_\alpha^\beta \mathbb{J}^\gamma \\
\{\mathbb{Q}_\alpha^a, \mathbb{Q}_\beta^b\} &= \epsilon_{\alpha\beta} \epsilon^{ab} \mathbb{C} & \{\mathbb{Q}_a^{\dagger\alpha}, \mathbb{Q}_b^{\dagger\beta}\} &= \epsilon^{\alpha\beta} \epsilon_{ab} \mathbb{C}^\dagger \\
\{\mathbb{Q}_\alpha^a, \mathbb{Q}_b^{\dagger\beta}\} &= \delta_b^a \mathbb{R}_\alpha^\beta + \delta_\alpha^\beta \mathbb{L}_b^a + \frac{1}{2} \delta_b^a \delta_\alpha^\beta \mathbb{H}.
\end{aligned} \tag{3.1}$$

The first two lines show how the indices of an arbitrary generator \mathbb{J} with the appropriate indices transform.

3.2 Fundamental Representation

The fundamental representation of \mathfrak{h} is four-dimensional. It is realized on a graded vector space with two bosonic basis vectors $|e_a\rangle$, $a = 1, 2$ and two fermionic basis vectors $|e_\alpha\rangle$, $\alpha = 1, 2$.

3.2.1 Matrix Realization

The two copies of $\mathfrak{su}(2)$ act canonically on both the bosonic and fermionic subspace

$$\mathbb{L}_b^a |e_c\rangle = \delta_c^a |e_b\rangle - \frac{1}{2} \delta_b^a |e_c\rangle \quad \mathbb{L}_b^a |e_\gamma\rangle = 0 \tag{3.2}$$

$$\mathbb{R}_\beta^\alpha |e_c\rangle = 0 \quad \mathbb{R}_\beta^\alpha |e_\gamma\rangle = \delta_\gamma^\alpha |e_\beta\rangle - \frac{1}{2} \delta_\beta^\alpha |e_\gamma\rangle. \tag{3.3}$$

The supercharges act as follows

$$\mathbb{Q}_\beta^a |e_c\rangle = a \delta_c^a |e_\beta\rangle \quad \mathbb{Q}_\beta^a |e_\gamma\rangle = b \epsilon_{\beta\gamma} \epsilon^{ab} |e_b\rangle \tag{3.4}$$

$$\mathbb{Q}_b^{\dagger\alpha} |e_c\rangle = c \epsilon^{\alpha\gamma} \epsilon_{bc} |e_\gamma\rangle \quad \mathbb{Q}_b^{\dagger\alpha} |e_\gamma\rangle = d \delta_\gamma^\alpha |e_b\rangle. \tag{3.5}$$

It is easily seen that all the defining commutation relations are satisfied, provided the parameters satisfy $ad - bc = 1$.

The values of the central charges are found by commuting the supercharges. They are all proportional to the identity matrix

$$\mathbb{H} = H \mathbb{1}, \quad \mathbb{C} = C \mathbb{1}, \quad \mathbb{C}^\dagger = C^\dagger \mathbb{1}. \tag{3.6}$$

and have eigenvalues

$$H = ad + bc, \quad C = ab, \quad C^\dagger = cd. \tag{3.7}$$

Because of the constraint $ad - bc = 1$, the central charges satisfy the ‘shortening’ condition

$$\mathbb{H}^2 - 4\mathbb{C}\mathbb{C}^\dagger = \mathbb{1}. \tag{3.8}$$

Since the central charges are related the representation is atypical; such a representation is called short [55]. In order for this representation to be unitary one needs

$$a = d^*, \quad b = c^*. \quad (3.9)$$

In unitary representations we find that \mathbb{C} is the Hermitian conjugate of \mathbb{C}^\dagger , which justifies our notation, and that the Hamiltonian \mathbb{H} is Hermitian and its eigenvalues are real.

For completeness, we explicitly list the matrix representations of a choice of simple roots of \mathfrak{h} :

$$\begin{aligned} \mathbb{L}_1^2 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \mathbb{L}_2^1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \mathbb{R}_3^4 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \mathbb{R}_4^3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ \mathbb{Q}_3^1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \mathbb{Q}_2^{\dagger 4} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d \\ c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (3.10)$$

The other elements are easily obtained by making use of the defining commutation relations.

3.2.2 Parameterizations

From the study of the light-cone gauged superstring on $\text{AdS}_5 \times \text{S}^5$ we know that the central charges are related to the world-sheet momentum p and the string tension g

$$\mathbb{C} = \frac{ig}{2}(e^{ip} - 1), \quad \mathbb{C}^\dagger = -\frac{ig}{2}(e^{-ip} - 1). \quad (3.11)$$

From this one can express the parameters a, b, c, d that describe the fundamental representation in terms of p, g . Introduce parameters x^\pm that are related to p, g by

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i}{g}, \quad \frac{x^+}{x^-} = e^{ip}. \quad (3.12)$$

Then we can write

$$\begin{aligned} a &= \sqrt{\frac{g}{2}}\eta, & b &= \sqrt{\frac{g}{2}}\frac{i}{\eta}\left(\frac{x^+}{x^-} - 1\right), \\ c &= -\sqrt{\frac{g}{2}}\frac{\eta}{x^+}, & d &= \sqrt{\frac{g}{2}}\frac{x^+}{i\eta}\left(1 - \frac{x^-}{x^+}\right). \end{aligned} \quad (3.13)$$

The parameter η corresponds to a rescaling of the bosonic basis vectors relative to the fermionic ones. However, upon insisting on unitarity of the representation it is easily seen that we are led to

$$\eta = e^{\frac{ip}{4}}\sqrt{ix^- - ix^+}. \quad (3.14)$$

The factor $e^{\frac{ip}{4}}$ is not a consequence of unitarity, but this particular choice will turn out to be convenient later on.

The central charges take the form

$$\begin{aligned} C &= \frac{g}{2i} \left(1 - \frac{x^+}{x^-} \right) = \frac{g}{2i} (1 - e^{ip}), & C^\dagger &= \frac{ig}{2} \left(1 - \frac{x^-}{x^+} \right) = \frac{ig}{2} (1 - e^{-ip}), \\ H &= ig \left(x^- - x^+ + \frac{i}{g} \right). \end{aligned} \quad (3.15)$$

From the shortening condition (3.8) one sees that the Hamiltonian satisfies a lattice type dispersion relation

$$H^2 = 1 + 4g^2 \sin^2(p/2). \quad (3.16)$$

It is also worthwhile mentioning that instead of p , one can define a rapidity variable z living on an elliptic curve (torus). Equation (3.16) can be uniformized on the torus, which prevents the appearance of branch cuts that arise when taking square roots in order to solve for one of the variables. In terms of Jacobi elliptic functions one finds

$$p = 2\text{am}z, \quad \sin \frac{p}{2} = 2\text{sn}(z, k), \quad H = 2\text{dn}(z, k), \quad (3.17)$$

where $k = -4g^2$. Written in terms of this rapidity variable the parameters x^\pm become

$$x^\pm = \frac{1 + \text{dn}z}{2g} \left(\frac{\text{cn}z}{\text{sn}z} \pm i \right). \quad (3.18)$$

The periods of the torus ω_1, ω_2 are defined by complete elliptic integrals of the first kind

$$2\omega_1 = 4K(k) \quad 2\omega_2 = 4iK(1-k) - 4K(k), \quad (3.19)$$

where

$$K(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{(2n-1)!!}{2n!!} \right]^2 k^{2n}. \quad (3.20)$$

3.3 The Outer Automorphism and $\mathfrak{gl}(2|2)$

The algebra \mathfrak{h} admits a useful family of outer automorphisms [55, 84] that form an $SL(2)$ group. It is defined by

$$\begin{pmatrix} \mathbb{Q}_\alpha^a \\ \epsilon^{ab} \epsilon_{\alpha\beta} \mathbb{Q}_b^{\dagger\beta} \end{pmatrix} \rightarrow \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} \mathbb{Q}_\alpha^a \\ \epsilon^{ab} \epsilon_{\alpha\beta} \mathbb{Q}_b^{\dagger\beta} \end{pmatrix}. \quad (3.21)$$

The parameters u_1, u_2, v_1, v_2 satisfy

$$u_1 v_2 - u_2 v_1 = 1. \quad (3.22)$$

This condition precisely defines an $SL(2)$ transformation. Under this automorphism the central charges transform as

$$\begin{pmatrix} \mathbb{H} \\ \mathbb{C} \\ \mathbb{C}^\dagger \end{pmatrix} \rightarrow \begin{pmatrix} (u_1 v_1 + u_2 v_2) \mathbb{H} + 2u_1 v_2 \mathbb{C} + 2u_2 v_1 \mathbb{C}^\dagger \\ u_1^2 \mathbb{C} + u_2^2 \mathbb{C}^\dagger + u_1 u_2 \mathbb{H} \\ v_1^2 \mathbb{C} + v_2^2 \mathbb{C}^\dagger + v_1 v_2 \mathbb{H} \end{pmatrix}. \quad (3.23)$$

It can be checked that the shortening condition (3.8) is invariant under this transformation.

A very useful application of this automorphism is that one can transform \mathfrak{h} into a normal $\mathfrak{su}(2|2)$ algebra and vice versa. Explicitly, the parameters that transform the central charges $\mathbb{C}, \mathbb{C}^\dagger$ to zero are given by

$$\begin{aligned} u_1 &= \frac{\mathbb{H} - \sqrt{\mathbb{H}^2 - 4\mathbb{C}\mathbb{C}^\dagger}}{2\mathbb{C}}, & u_2 &= -1, \\ v_1 &= \frac{1}{2} \left(1 + \frac{\mathbb{H}}{\sqrt{\mathbb{H}^2 - 4\mathbb{C}\mathbb{C}^\dagger}} \right), & v_2 &= -\frac{\mathbb{C}}{\sqrt{\mathbb{H}^2 - 4\mathbb{C}\mathbb{C}^\dagger}}. \end{aligned} \quad (3.24)$$

Notice the appearance of the shortening condition (3.8). A special case of this automorphism occurs when $u_2 = 0 = v_1$ and $u_1 = e^{i\phi}$ for some real phase ϕ .

$$\mathbb{Q}_\alpha^a \rightarrow e^{i\phi} \mathbb{Q}_\alpha^a \quad \mathbb{Q}_a^{\dagger\alpha} \rightarrow e^{-i\phi} \mathbb{Q}_a^{\dagger\alpha} \quad \mathbb{C} \rightarrow e^{2i\phi} \mathbb{C} \quad \mathbb{C}^\dagger \rightarrow e^{-2i\phi} \mathbb{C}^\dagger. \quad (3.25)$$

Relation to $\mathfrak{gl}(2|2)$

By the $SL(2)$ automorphism one can now transform any representation of the bigger Lie superalgebra $\mathfrak{gl}(2|2)$ into a representation of \mathfrak{h} . This is convenient, since in the paper [123] all finite-dimensional irreducible representations of $\mathfrak{gl}(2|2)$ are explicitly constructed in an oscillator basis. Generators of $\mathfrak{gl}(2|2)$ are denoted by E_{ij} , with commutation relations

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - (-)^{(d[i]+d[j])(d[k]+d[l])} \delta_{il} E_{kj}. \quad (3.26)$$

Indices i, j, k, l run from 1 to 4, and the fermionic grading is assigned as $d[1] = d[2] = 0$, $d[3] = d[4] = 1$. The quadratic Casimir of this algebra is $C_2 = \sum_{i,j=1}^4 (-)^{d[j]} E_{ij} E_{ji}$. One finds that the finite dimensional irreps are labelled by two half-integers $j_1, j_2 = 0, \frac{1}{2}, \dots$, and two complex numbers q and y . These numbers correspond to the values taken by appropriate generators on the highest weight state $|\omega\rangle$ of the representation, defined by the following conditions:

$$\begin{aligned} H_1 |\omega\rangle &= (E_{11} - E_{22}) |\omega\rangle = 2j_1 |\omega\rangle, & H_2 |\omega\rangle &= (E_{33} - E_{44}) |\omega\rangle = 2j_2 |\omega\rangle, \\ I |\omega\rangle &= \sum_{i=1}^4 E_{ii} |\omega\rangle = 2q |\omega\rangle, & N |\omega\rangle &= \sum_{i=1}^4 (-)^{[i]} E_{ii} |\omega\rangle = 2y |\omega\rangle, & E_{i<j} |\omega\rangle &= 0. \end{aligned} \quad (3.27)$$

The generator N never appears on the right hand side of the commutation relations, therefore it is defined up to the addition of a central element βI , with β a constant¹. This also means

¹We drop the term βI since it will not affect our discussion.

that we can consistently mod out the generator N , and obtain $\mathfrak{sl}(2|2)$ as a subalgebra of the original $\mathfrak{gl}(2|2)$ algebra². In order to construct representations of the centrally-extended $\mathfrak{su}(2|2)$ Lie superalgebra, we then first mod out N , and subsequently perform an $\mathfrak{sl}(2)$ rotation by means of the outer automorphism (3.21).

The way the outer automorphism is implemented is by mapping the $\mathfrak{gl}(2|2)$ non-diagonal generators into new generators as follows:

$$\begin{aligned}\mathbb{L}_a^b &= E_{ab} \quad \forall a \neq b, & \mathbb{R}_\alpha^\beta &= E_{\alpha\beta} \quad \forall \alpha \neq \beta, \\ \mathbb{Q}_\alpha^a &= a E_{\alpha a} + b \epsilon_{\alpha\beta} \epsilon^{ab} E_{b\beta}, \\ \mathbb{Q}_a^{\dagger\alpha} &= c \epsilon_{ab} \epsilon^{\alpha\beta} E_{\beta b} + d E_{a\alpha},\end{aligned}\tag{3.28}$$

subject to the constraint

$$ad - bc = 1.\tag{3.29}$$

The diagonal elements are automatically obtained by commuting positive and negative roots. In particular, one obtains the following values of the central charges:

$$\mathbb{H} = (ad + bc) I, \quad \mathbb{C} = ab I, \quad \mathbb{C}^\dagger = cd I.\tag{3.30}$$

Note that in [123] I is just proportional to the identity operator $I = 2q\mathbb{1}$. Moreover, it turns out that the generator N will play an important role later on. Let us discuss its properties here. We define the following operator

$$\mathbb{B} = \frac{1}{2} \frac{1}{ad + bc} N,\tag{3.31}$$

which satisfies the following commutation relations

$$\begin{aligned}[\mathbb{B}, \mathbb{Q}_\alpha^a] &= -\mathbb{Q}_\alpha^a + 2\mathbb{C}\mathbb{H}^{-1} \epsilon_{\alpha\beta} \epsilon^{ab} \mathbb{Q}_b^{\dagger\alpha} \\ [\mathbb{B}, \mathbb{Q}_a^{\dagger\alpha}] &= \mathbb{Q}_a^{\dagger\alpha} - 2\mathbb{C}^\dagger \mathbb{H}^{-1} \epsilon^{\alpha\beta} \epsilon_{ab} \mathbb{Q}_\beta^b \\ [\mathbb{B}, \mathbb{L}_b^a] &= [\mathbb{B}, \mathbb{R}_\beta^\alpha] = [\mathbb{B}, \mathbb{H}] = 0.\end{aligned}\tag{3.32}$$

Notice that since central charges are actually proportional to I we have replaced terms like $\frac{ab}{ad+bc}$ by $\mathbb{C}\mathbb{H}^{-1}$. Furthermore, by defining the quadratic operator

$$\mathcal{T} = \mathbb{R}_\beta^\alpha \mathbb{R}_\alpha^\beta - \mathbb{L}_b^a \mathbb{L}_a^b + \mathbb{Q}_a^{\dagger\alpha} \mathbb{Q}_\alpha^a - \mathbb{Q}_\alpha^a \mathbb{Q}_a^{\dagger\alpha},\tag{3.33}$$

it follows that \mathbb{B} and \mathcal{T} can be used to construct a generalized Casimir operator C_2

$$C_2 = \mathbb{B}\mathbb{H} - \mathcal{T}.\tag{3.34}$$

²Further modding out of the center I produces the simple Lie superalgebra $\mathfrak{psl}(2|2)$. The representation theory of $\mathfrak{psl}(2|2)$ has been completely classified in [124].

Generalized here means that this operator can be shown to be central upon using the non-linear commutation relations (3.32). Hence if one constructs a representation of \mathfrak{h} by using the $\mathfrak{sl}(2)$ rotation procedure we have described before, one can supply it with an extra generator \mathbb{B} . This generator would be the missing element to complete the algebra to $\mathfrak{gl}(2|2)$, if it were not for its non-linear commutation relations.

3.4 Symmetric Short Representations

An important class of representations are the symmetric short representations. It turns out that they describe bound states of $\text{AdS}_5 \times \text{S}^5$ world sheet excitations. These representations are indexed by a positive integer ℓ and the corresponding representation is 4ℓ dimensional. This 4ℓ dimensional atypical symmetric representation is realized on a graded vector space with basis $|e_{a_1 \dots a_\ell}\rangle$, $|e_{a_1 \dots a_{\ell-1} \alpha}\rangle$ and $|e_{a_1 \dots a_{\ell-2} \alpha \beta}\rangle$, where a_i are bosonic indices and α, β are fermionic indices. Each of the basis vectors is totally symmetric in the bosonic indices and anti-symmetric in the fermionic indices [52–55, 84]. We will refer to these representations as bound state representations.

The most convenient way to describe these representations is by the so-called superspace formalism, introduced in [84]. In this formalism the basis vectors correspond to monomials and the algebra generators are differential operators. The big advantage of this formalism is that it allows one to treat all bound states at once instead of dealing with matrices of arbitrary (big) size 4ℓ .

Consider the vector space of analytic functions of two bosonic variables $w_{1,2}$ and two fermionic variables $\theta_{3,4}$. Since we are dealing with analytic functions we can expand any such function $\Phi(w, \theta)$:

$$\begin{aligned} \Phi(w, \theta) &= \sum_{\ell=0}^{\infty} \Phi_\ell(w, \theta), \\ \Phi_\ell &= \phi^{a_1 \dots a_\ell} w_{a_1} \dots w_{a_\ell} + \phi^{a_1 \dots a_{\ell-1} \alpha} w_{a_1} \dots w_{a_{\ell-1}} \theta_\alpha + \\ &\quad + \phi^{a_1 \dots a_{\ell-2} \alpha \beta} w_{a_1} \dots w_{a_{\ell-2}} \theta_\alpha \theta_\beta. \end{aligned} \tag{3.35}$$

In terms of the above analytic functions, the basis vectors of the totally symmetric representation can clearly be identified as $|e_{a_1 \dots a_\ell}\rangle \leftrightarrow w_{a_1} \dots w_{a_\ell}$, $|e_{a_1 \dots a_{\ell-1} \alpha}\rangle \leftrightarrow w_{a_1} \dots w_{a_{\ell-1}} \theta_\alpha$ and $|e_{a_1 \dots a_{\ell-2} \alpha \beta}\rangle \leftrightarrow w_{a_1} \dots w_{a_{\ell-2}} \theta_\alpha \theta_\beta$, respectively. In other words, we find the atypical totally symmetric representation of dimension 4ℓ when we restrict to terms Φ_ℓ , i.e. monomials of degree ℓ .

In this representation the algebra generators can be written in differential operator form as

$$\mathbb{L}_a^b = w_a \frac{\partial}{\partial w_b} - \frac{1}{2} \delta_a^b w_c \frac{\partial}{\partial w_c}, \quad \mathbb{R}_\alpha^\beta = \theta_\alpha \frac{\partial}{\partial \theta_\beta} - \frac{1}{2} \delta_\alpha^\beta \theta_\gamma \frac{\partial}{\partial \theta_\gamma}, \quad (3.36)$$

$$\mathbb{Q}_\alpha^a = a \theta_\alpha \frac{\partial}{\partial w_a} + b \epsilon^{ab} \epsilon_{\alpha\beta} w_b \frac{\partial}{\partial \theta_\beta}, \quad \mathbb{Q}_a^{\dagger\alpha} = d w_a \frac{\partial}{\partial \theta_\alpha} + c \epsilon_{ab} \epsilon^{\alpha\beta} \theta_\beta \frac{\partial}{\partial w_b}, \quad (3.37)$$

and the central charges are

$$\mathbb{C} = ab \left(w_a \frac{\partial}{\partial w_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha} \right), \quad \mathbb{C}^\dagger = cd \left(w_a \frac{\partial}{\partial w_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha} \right), \quad (3.38)$$

$$\mathbb{H} = (ad + bc) \left(w_a \frac{\partial}{\partial w_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha} \right). \quad (3.39)$$

To form a representation, the parameters a, b, c, d again must satisfy the condition $ad - bc = 1$. The central charges become ℓ dependent:

$$H = \ell(ad + bc), \quad C = \ell ab, \quad C^\dagger = \ell cd. \quad (3.40)$$

The parameters a, b, c, d can be expressed in terms of the bound state momentum p and the coupling g :

$$\begin{aligned} a &= \sqrt{\frac{g}{2\ell}} \eta, & b &= \sqrt{\frac{g}{2\ell}} \frac{i}{\eta} \left(\frac{x^+}{x^-} - 1 \right), \\ c &= -\sqrt{\frac{g}{2\ell}} \frac{\eta}{x^+}, & d &= \sqrt{\frac{g}{2\ell}} \frac{x^+}{i\eta} \left(1 - \frac{x^-}{x^+} \right), \end{aligned} \quad (3.41)$$

where the parameters x^\pm satisfy

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i\ell}{g}, \quad \frac{x^+}{x^-} = e^{ip} \quad (3.42)$$

and the parameter η is given by

$$\eta = \eta(p), \quad \eta(p) = e^{i\frac{p}{4}} \sqrt{ix^- - ix^+}. \quad (3.43)$$

The fundamental representation is obtained by taking $\ell = 1$.

Tensor product representations are easily obtained by multiplying these superfields. Note that due to the fermionic nature of the θ variables we are automatically dealing with graded tensor products. For example the tensor product of two fundamental representations is described by monomials

$$\{w_a v_b, w_a \vartheta_\beta, \theta_\alpha v_b, \theta_\alpha \vartheta_\beta\}, \quad (3.44)$$

where the variables w, θ and v, ϑ describe the first and the second fundamental representation respectively.

These bound state representations can be obtained starting from a $\mathfrak{gl}(2|2)$ representation by identifying $E_{ab} = w_a \partial_{w_b}, \dots$ and proceeding as described in section 3.3. This means that one can find an analog of (3.31) in these representations

$$\mathbb{B} = \frac{1}{2(ad + bc)} \left(w_a \frac{\partial}{\partial w_a} - \theta_\alpha \frac{\partial}{\partial \theta_\alpha} \right). \quad (3.45)$$

The quadratic Casimir takes values

$$C_2 = \ell(\ell - 1)\mathbb{1} \quad (3.46)$$

and is indeed central.

3.5 Hopf Algebra Structure

In order to have a consistent Hopf algebra structure for \mathfrak{h} and for its Yangian which we introduce in the next section, one needs to consider a modified coproduct structure. To this end we introduce an additional central generator \mathbb{U} , which is closely related to the central charges. Let us equip the symmetry algebra with the following deformed Hopf-algebra (opposite) coproduct [125, 126]

$$\begin{aligned} \Delta(\mathbb{J}) &= \mathbb{J} \otimes \mathbb{U}^{[[\mathbb{J}]]} + \mathbb{1} \otimes \mathbb{J}, & \Delta^{op}(\mathbb{J}) &= \mathbb{J} \otimes \mathbb{1} + \mathbb{U}^{[[\mathbb{J}]]} \otimes \mathbb{J}, \\ \Delta(\mathbb{U}) &= \mathbb{U} \otimes \mathbb{U}, & \Delta^{op}(\mathbb{U}) &= \mathbb{U} \otimes \mathbb{U}, \end{aligned} \quad (3.47)$$

where \mathbb{J} is any generator of \mathfrak{h} , $[[\mathbb{J}]] = 0$ for the bosonic $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ generators and for the energy generator \mathbb{H} , $[[\mathbb{J}]] = 1$ (resp., -1) for the \mathbb{Q} (resp., \mathbb{Q}^\dagger) supercharges, and $[[\mathbb{J}]] = 2$ (resp., -2) for the central charge \mathbb{C} (resp. \mathbb{C}^\dagger). The fact that $[[\mathbb{C}]] = 2$, even though it is central like the Hamiltonian, is a consequence of Δ respecting the Lie bracket. The value of \mathbb{U} is determined by the consistency requirement that the coproduct is cocommutative on the center. Since the S-matrix should commute with the center, one finds that this is a necessary condition for the existence of an S-matrix (2.28). This produces the algebraic condition

$$\mathbb{U}^2 = \kappa \mathbb{C} + \mathbb{1} \quad (3.48)$$

for some representation-independent constant κ . With our choice of parameterization of the central charge for short representations (3.15) it follows that $\kappa = \frac{2}{ig}$, and we obtain the relation

$$\mathbb{U} = \sqrt{\frac{x^+}{x^-}} \mathbb{1} = e^{i\frac{p}{2}} \mathbb{1}. \quad (3.49)$$

We call \mathbb{U} a braiding factor. In order to have a complete realization of the Hopf algebra, one needs to specify the antipode map \mathcal{S} [41, 84, 126] and the co-unit

$$\epsilon(\mathbb{1}) = 1, \quad \epsilon(\mathbb{J}) = 0, \quad \epsilon(\mathbb{U}) = 1. \quad (3.50)$$

From (2.25) one deduces that the inclusion of \mathbb{U} also alters the antipode which becomes

$$\mathcal{S}(\mathbb{J}) = -\mathbb{U}^{-[[\mathbb{J}]]}\mathbb{J}. \quad (3.51)$$

The antipode map corresponds to particle to anti-particle transformations, and has an alternative description in terms of a charge conjugation matrix \mathcal{C} as we will now explain. The z -torus offers a convenient way to describe anti-particles. One finds that

$$H(z + \omega_2) = -H(z) \quad p(z + \omega_2) = -p(z). \quad (3.52)$$

This is similar to the crossing transformation in relativistic models. On the level of the parameters x^\pm this crossing transformation is

$$x^\pm \rightarrow \frac{1}{x^\pm}. \quad (3.53)$$

Consider the map

$$\mathbb{J} \rightarrow -\mathbb{J}^{st}, \quad (3.54)$$

which preserves the $\mathfrak{su}(2|2)$ commutation relations. Letting this map act on an irrep of centrally $\mathfrak{su}(2|2)$ with central elements \mathbb{H}, \mathbb{C} clearly gives a different irrep of centrally extended $\mathfrak{su}(2|2)$ with central elements $-\mathbb{H}, -\mathbb{C}$. Under the crossing relation the central charge \mathbb{C} transforms as

$$\mathbb{C}(z + \omega_2) = -e^{-ip}\mathbb{C}(z). \quad (3.55)$$

The phase e^{-ip} can be absorbed by the $U(1)$ -automorphism (3.25), i.e. we choose the phase in this automorphism to be $e^{\frac{ip}{2}} = \mathbb{U}$. We see that for short representations acting with the antipode on the algebra generators produces the same set of central charges as the above described anti-particle transformation. This indicates that there should be a similarity transformation by which we can relate the two

$$\mathcal{S}(\mathbb{J}) = -\mathbb{U}^{-[[\mathbb{J}]]}\mathbb{J} = \mathcal{C}\bar{\mathbb{J}}^{st}\mathcal{C}^{-1}, \quad (3.56)$$

where $\bar{\mathbb{J}} = \mathbb{J}(z + \omega_2)$. This transformation matrix is called the charge conjugation matrix \mathcal{C} . One finds for the fundamental representation it is given by

$$\mathcal{C} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad (3.57)$$

For generic bound state representations in the operator language, the conjugation operator is

$$\mathcal{C} = -i\epsilon^{ab}w_a\partial_{w_b} + \epsilon^{\alpha\beta}\theta_\alpha\partial_{\theta_\beta}. \quad (3.58)$$

On the uniformizing torus, applying the particle to anti-particle transformation four times gives the identity.

It is readily checked that the structure we just introduced satisfies all the defining relations of Hopf algebras. In particular, the braided (opposite) coproduct respects the Lie bracket

$$\Delta[\mathbb{J}^A, \mathbb{J}^B] = [\Delta\mathbb{J}^A, \Delta\mathbb{J}^B]. \quad (3.59)$$

Here we can see the convenience of our particular choice of η (3.14). It turns out that for this choice

$$(\Delta\mathbb{L}_b^a)^t = \Delta^{op}\mathbb{L}_a^b \quad (\Delta\mathbb{R}_\beta^\alpha)^t = \Delta^{op}\mathbb{L}_\alpha^\beta \quad (3.60)$$

$$(\Delta\mathbb{Q}_\alpha^a)^t \sim \Delta^{op}\mathbb{Q}_a^{\dagger\alpha} \quad (\Delta\mathbb{Q}_a^{\dagger\alpha})^t \sim \Delta^{op}\mathbb{Q}_\alpha^a. \quad (3.61)$$

This means that the R-matrix will be symmetric.

Finally, we would like to discuss a more technical rewriting of the braided structure. It is worthwhile to notice that the braiding factors appearing in the coproducts can be absorbed explicitly in the parameters a, b, c, d . Explicitly, the parameters for the tensor products of two bound state representations appearing in the coproduct (3.47) are given by:

$$\begin{aligned} a_1 &= \sqrt{\frac{g}{2\ell_1}}\eta_1, & b_1 &= -ie^{ip_2}\sqrt{\frac{g}{2\ell_1}}\frac{1}{\eta_1}\left(\frac{x_1^+}{x_1^-} - 1\right), \\ c_1 &= -e^{-ip_2}\sqrt{\frac{g}{2\ell_1}}\frac{\eta_1}{x_1^+}, & d_1 &= i\sqrt{\frac{g}{2\ell_1}}\frac{x_1^+}{\eta_1}\left(\frac{x_1^-}{x_1^+} - 1\right), \\ \eta_1 &= e^{i\frac{p_1}{4}}e^{i\frac{p_2}{2}}\sqrt{ix_1^- - ix_1^+}, & & \\ & & & (3.62) \end{aligned}$$

$$\begin{aligned} a_2 &= \sqrt{\frac{g}{2\ell_2}}\eta_2, & b_2 &= -i\sqrt{\frac{g}{2\ell_2}}\frac{1}{\eta_2}\left(\frac{x_2^+}{x_2^-} - 1\right), \\ c_2 &= -\sqrt{\frac{g}{2\ell_2}}\frac{\eta_2}{x_2^+}, & d_2 &= i\sqrt{\frac{g}{2\ell_2}}\frac{x_2^+}{\eta_2}\left(\frac{x_2^-}{x_2^+} - 1\right), \\ \eta_2 &= e^{i\frac{p_2}{4}}\sqrt{ix_2^- - ix_2^+}, & & \end{aligned}$$

where the indices 1, 2 refer to first and second space respectively. One sees now that the effect of the braiding factor \mathbb{U} causes the parameters of the first space to depend on the momentum p_2 of the second particle.

Accordingly, the labels used in Δ^{op} are given by (we supply them with indices 3, 4 to make notational distinction between opposite and normal coproduct):

$$\begin{aligned} a_3 &= \sqrt{\frac{g}{2\ell_1}}\eta_1^{op}, & b_3 &= -i\sqrt{\frac{g}{2\ell_1}}\frac{1}{\eta_1^{op}}\left(\frac{x_1^+}{x_1^-} - 1\right), \\ c_3 &= -\sqrt{\frac{g}{2\ell_1}}\frac{\eta_1^{op}}{x_1^+}, & d_3 &= i\sqrt{\frac{g}{2\ell_1}}\frac{x_1^+}{\eta_1^{op}}\left(\frac{x_1^-}{x_1^+} - 1\right), \\ \eta_1^{op} &= e^{i\frac{p_1}{4}}\sqrt{ix_1^- - ix_1^+}, & & \\ & & & (3.63) \end{aligned}$$

$$\begin{aligned} a_4 &= \sqrt{\frac{g}{2\ell_2}}\eta_2^{op}, & b_4 &= -ie^{ip_1}\sqrt{\frac{g}{2\ell_2}}\frac{1}{\eta_2^{op}}\left(\frac{x_2^+}{x_2^-} - 1\right), \\ c_4 &= -e^{-ip_1}\sqrt{\frac{g}{2\ell_2}}\frac{\eta_2^{op}}{x_2^+}, & d_4 &= i\sqrt{\frac{g}{2\ell_2}}\frac{x_2^+}{\eta_2^{op}}\left(\frac{x_2^-}{x_2^+} - 1\right), \\ \eta_2^{op} &= e^{i\frac{p_2}{4}}e^{i\frac{p_1}{2}}\sqrt{ix_2^- - ix_2^+}. \end{aligned}$$

When using this parameters these parameters the coproduct looks standard again

$$\Delta \mathbb{J}^A = \mathbb{J}^A(a_1, b_1, c_1, d_1) \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{J}^A(a_2, b_2, c_2, d_2) \quad (3.64)$$

$$\Delta^{op} \mathbb{J}^A = \mathbb{J}^A(a_3, b_3, c_3, d_3) \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{J}^A(a_4, b_4, c_4, d_4). \quad (3.65)$$

The non-trivial braiding factors are all hidden in the parameters of the four representations involved. The reason for this technical excursion is that it will make future computations and results more transparent.

3.6 The Yangian of centrally extended $\mathfrak{su}(2|2)$

Next we discuss the Yangian of \mathfrak{h} . One can check that the Cartan matrix of \mathfrak{h} is not invertible. Actually one finds that this algebra also does not allow for a non-zero bilinear (Killing) form. This indicates that one cannot straightforwardly apply formula (2.33) since we have no means of lowering the Lie algebra indices with the Killing form. Nevertheless, it turns out that $\mathfrak{su}(2|2)$ admits a Yangian. The Yangian structure is somewhat unconventional since in evaluation representations (2.38) quasi-cocommutativity implies a relation between the evaluation parameter u and the parameters of the representation as we will explain later on.

3.6.1 First realization

Let us introduce an additional set of generators $\hat{\mathbb{J}}^A$ that satisfy

$$[\mathbb{J}^A, \hat{\mathbb{J}}^B] = f_C^{AB} \hat{\mathbb{J}}^C, \quad (3.66)$$

where f_C^{AB} are the structure constants of \mathfrak{h} . The algebra generated by these generators, together with the generators \mathbb{J}^A of \mathfrak{h} is called the Yangian $Y(\mathfrak{h})$ of \mathfrak{h} , see section 2.4.

The absence of a non-zero Killing form prevents one from using (2.33) to find the Hopf algebra structure of $Y(\mathfrak{h})$. In order to be able to still derive coproducts of the Yangian type (2.33) one can apply a number of techniques, for example one can make use of a limiting procedure on the exceptional algebra $D(2, 1; \varepsilon)$ [127]. This is done in Appendix A of this chapter. A different approach was followed in [83] where automorphisms were used. Of course, these procedures are not direct computations of the coproduct structure and one has to check afterwards whether the found coproduct satisfies all the defining relations. Let us list the explicit formulae for the

coproducts here

$$\Delta(\hat{\mathbb{L}}_b^a) = \hat{\mathbb{L}}_b^a \otimes 1 + 1 \otimes \hat{\mathbb{L}}_b^a + \frac{1}{2} \left[\mathbb{L}_b^c \otimes \mathbb{L}_c^a - \mathbb{L}_c^a \otimes \mathbb{L}_b^c + \right. \\ \left. - \mathbb{Q}_b^{\dagger\gamma} \otimes \mathbb{U}^{-1} \mathbb{Q}_\gamma^a - \mathbb{Q}_\gamma^a \otimes \mathbb{U} \mathbb{Q}_b^{\dagger\gamma} + \frac{\delta_b^a}{2} (\mathbb{Q}_c^{\dagger\gamma} \otimes \mathbb{U}^{-1} \mathbb{Q}_\gamma^c + \mathbb{Q}_\gamma^c \otimes \mathbb{U} \mathbb{Q}_c^{\dagger\gamma}) \right] \quad (3.67)$$

$$\Delta(\hat{\mathbb{R}}_\beta^\alpha) = \hat{\mathbb{R}}_\beta^\alpha \otimes 1 + 1 \otimes \hat{\mathbb{R}}_\beta^\alpha + \frac{1}{2} \left[-\mathbb{R}_\beta^\gamma \otimes \mathbb{R}_\gamma^\alpha + \mathbb{R}_\gamma^\alpha \otimes \mathbb{R}_\beta^\gamma + \right. \\ \left. + \mathbb{Q}_c^{\dagger\alpha} \otimes \mathbb{U}^{-1} \mathbb{Q}_\beta^c + \mathbb{Q}_\beta^c \otimes \mathbb{U} \mathbb{Q}_c^{\dagger\alpha} - \frac{\delta_\beta^\alpha}{2} (\mathbb{Q}_c^{\dagger\gamma} \otimes \mathbb{U}^{-1} \mathbb{Q}_\gamma^c + \mathbb{Q}_\gamma^c \otimes \mathbb{U} \mathbb{Q}_c^{\dagger\gamma}) \right] \quad (3.68)$$

$$\Delta(\hat{\mathbb{Q}}_\alpha^a) = \hat{\mathbb{Q}}_\alpha^a \otimes \mathbb{U} + 1 \otimes \hat{\mathbb{Q}}_\alpha^a + \frac{1}{2} \left[-\mathbb{R}_\alpha^\gamma \otimes \mathbb{Q}_\gamma^a + \mathbb{Q}_\gamma^a \otimes \mathbb{U} \mathbb{R}_\alpha^\gamma + \right. \\ \left. - \mathbb{L}_{1;c}^a \otimes \mathbb{Q}_\alpha^c + \mathbb{Q}_\alpha^c \otimes \mathbb{U} \mathbb{L}_c^a - \frac{1}{2} \mathbb{H}_1 \otimes \mathbb{Q}_\alpha^a + \frac{1}{2} \mathbb{Q}_\alpha^a \otimes \mathbb{U} \mathbb{H}_1 + \right. \\ \left. + \epsilon_{\alpha\gamma} \epsilon^{ac} \mathbb{C} \otimes \mathbb{U}^2 \mathbb{Q}_c^{\dagger\gamma} - \epsilon_{\alpha\gamma} \epsilon^{ac} \mathbb{Q}_c^{\dagger\gamma} \otimes \mathbb{U}^{-1} \mathbb{C} \right], \quad (3.69)$$

$$\Delta(\hat{\mathbb{Q}}_a^{\dagger\alpha}) = \hat{\mathbb{Q}}_a^{\dagger\alpha} \otimes \mathbb{U}^{-1} + 1 \otimes \hat{\mathbb{Q}}_a^{\dagger\alpha} + \frac{1}{2} \left[\mathbb{L}_a^c \otimes \mathbb{Q}_a^{\dagger\alpha} - \mathbb{Q}_c^{\dagger\alpha} \otimes \mathbb{U}^{-1} \mathbb{L}_a^c + \right. \\ \left. + \mathbb{R}_\gamma^\alpha \otimes \mathbb{Q}_a^{\dagger\gamma} - \mathbb{Q}_a^{\dagger\gamma} \otimes \mathbb{U}^{-1} \mathbb{R}_\gamma^\alpha + \frac{1}{2} \mathbb{H} \otimes \mathbb{Q}_a^{\dagger\alpha} - \frac{1}{2} \mathbb{Q}_a^{\dagger\alpha} \otimes \mathbb{U}^{-1} \mathbb{H} + \right. \\ \left. - \epsilon_{ac} \epsilon^{\alpha\gamma} \mathbb{C}^\dagger \otimes \mathbb{U}^{-2} \mathbb{Q}_\gamma^c + \epsilon_{ac} \epsilon^{\alpha\gamma} \mathbb{Q}_\gamma^c \otimes \mathbb{U} \mathbb{C}^\dagger \right]. \quad (3.70)$$

and central charges

$$\Delta(\hat{\mathbb{H}}) = \hat{\mathbb{H}} \otimes 1 + 1 \otimes \hat{\mathbb{H}} + \mathbb{C} \otimes \mathbb{U}^2 \mathbb{C}^\dagger - \mathbb{C}^\dagger \otimes \mathbb{U}^{-2} \mathbb{C}, \quad (3.71)$$

$$\Delta(\hat{\mathbb{C}}) = \hat{\mathbb{C}} \otimes \mathbb{U}^2 + 1 \otimes \hat{\mathbb{C}} + \frac{1}{2} [\mathbb{H} \otimes \mathbb{C} - \mathbb{C} \otimes \mathbb{U}^2 \mathbb{H}], \quad (3.72)$$

$$\Delta(\hat{\mathbb{C}}^\dagger) = \hat{\mathbb{C}}^\dagger \otimes \mathbb{U}^{-2} + 1 \otimes \hat{\mathbb{C}}^\dagger - \frac{1}{2} [\mathbb{H} \otimes \mathbb{C}^\dagger - \mathbb{C}^\dagger \otimes \mathbb{U}^{-2} \mathbb{H}]. \quad (3.73)$$

It is indeed readily seen that the above introduced coproduct respects the commutator structure of the Yangian. Finally, to complete the Hopf algebra structure, we give the antipode and the counit

$$\mathcal{S}(\hat{\mathbb{J}}^A) = -\mathbb{U}^{-[[A]]} \hat{\mathbb{J}}^A, \quad \epsilon(\hat{\mathbb{J}}^A) = 0. \quad (3.74)$$

The reason the antipode does not have the extra (structure constant dependent) term as in (2.34) is because of the vanishing of the Killing form.

In order for a evaluation type representation (2.38) to be quasi-cocommutative we need to have that $\Delta \hat{\mathbb{C}} = \Delta^{op} \hat{\mathbb{C}}$ because this element is central. Recall that the central charge is related

to the braiding factor via (3.48). From this we derive

$$\begin{aligned} 0 &= \Delta \hat{\mathbb{C}} - \Delta^{op} \hat{\mathbb{C}} \\ &= \frac{ig}{2} \left[u_1(\mathbb{U}^2 - 1) - \frac{\mathbb{H}(\mathbb{U}^2 + 1)}{2} \right] \otimes (\mathbb{U}^2 - 1) \\ &\quad + (\mathbb{U}^2 - 1) \otimes \frac{ig}{2} \left[u_2(\mathbb{U}^2 - 1) - \frac{\mathbb{H}(\mathbb{U}^2 + 1)}{2} \right]. \end{aligned} \quad (3.75)$$

This implies

$$u = \frac{\mathbb{H} \mathbb{U}^2 + 1}{2 \mathbb{U}^2 - 1}. \quad (3.76)$$

Notice that the non-triviality of the braiding factor $\mathbb{U} \neq 1$ is important here. Strictly speaking one can also add a representation independent constant to u . However, it will be clear from the explicit derivations in later chapters that this constant does not enter the discussion and for that reason we set it to 0. In our explicit parameterization in terms of x^\pm (3.12) the evaluation parameter u becomes

$$u = \frac{g}{4i}(x^+ + x^-) \left(1 + \frac{1}{x^+ x^-} \right). \quad (3.77)$$

This feature is rather unusual since in most models the evaluation parameter u is unrelated to the representation of the Lie algebra.

In each of the representations of $\mathfrak{su}(2|2)$ discussed above one can consider the evaluation representation of the Yangian. However, explicitly checking the Serre identities proves to be difficult and becomes more transparent in Drinfeld's second realization.

3.6.2 Second realization

Let us continue with the discussion of Drinfeld's second realization [114]. As already clear from the previous discussion, also in this case the peculiar features of \mathfrak{h} make the analysis more complicated than for standard Lie superalgebras.

Let us first indicate the Chevalley-Serre generators

$$\mathfrak{E}_1^+ = \mathbb{Q}_2^{\dagger 4}, \quad \mathfrak{E}_1^- = \mathbb{Q}_4^2, \quad \mathfrak{H}_1 = -\mathbb{L}_1^1 - \mathbb{R}_3^3 + \frac{1}{2}\mathbb{H}, \quad (3.78)$$

$$\mathfrak{E}_2^+ = i\mathbb{Q}_4^1, \quad \mathfrak{E}_2^- = i\mathbb{Q}_1^{\dagger 4}, \quad \mathfrak{H}_2 = -\mathbb{L}_1^1 + \mathbb{R}_3^3 - \frac{1}{2}\mathbb{H}, \quad (3.79)$$

$$\mathfrak{E}_3^+ = i\mathbb{Q}_3^2, \quad \mathfrak{E}_3^- = i\mathbb{Q}_2^{\dagger 3}, \quad \mathfrak{H}_3 = \mathbb{L}_1^1 - \mathbb{R}_3^3 - \frac{1}{2}\mathbb{H}. \quad (3.80)$$

They satisfy the defining relations

$$[\mathfrak{H}_i, \mathfrak{H}_j] = 0, \quad [\mathfrak{H}_i, \mathfrak{E}_j^\pm] = \pm a_{ij} \mathfrak{E}_j^\pm, \quad \{\mathfrak{E}_i^+, \mathfrak{E}_j^-\} = \delta_{ij} \mathfrak{H}_i, \quad (3.81)$$

and the Serre relations

$$\text{ad}(\mathfrak{E}_1^\pm)^2(\mathfrak{E}_2^\pm) = \text{ad}(\mathfrak{E}_2^\pm)^2(\mathfrak{E}_1^\pm) = 0, \quad \{\mathfrak{E}_2^\pm, \mathfrak{E}_3^\pm\} = \text{central}, \quad (3.82)$$

with Cartan matrix

$$a_{ij} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \quad (3.83)$$

Notice that the Cartan matrix is degenerate.

Drinfeld's second realization [128] is now expressed in terms of Cartan generators $\kappa_{i,m}$ and fermionic simple roots $\xi_{i,m}^\pm$, $i = 1, 2, 3$, $m = 0, 1, 2, \dots$, subject to the following relations:

$$\begin{aligned} [\kappa_{i,m}, \kappa_{j,n}] &= 0, \quad [\kappa_{i,0}, \xi_{j,m}^+] = a_{ij} \xi_{j,m}^+, \\ [\kappa_{i,0}, \xi_{j,m}^-] &= -a_{ij} \xi_{j,m}^-, \quad \{\xi_{i,m}^+, \xi_{j,n}^-\} = \delta_{i,j} \kappa_{j,n+m}, \\ [\kappa_{i,m+1}, \xi_{j,n}^+] - [\kappa_{i,m}, \xi_{j,n+1}^+] &= \frac{1}{2} a_{ij} \{\kappa_{i,m}, \xi_{j,n}^+\}, \\ [\kappa_{i,m+1}, \xi_{j,n}^-] - [\kappa_{i,m}, \xi_{j,n+1}^-] &= -\frac{1}{2} a_{ij} \{\kappa_{i,m}, \xi_{j,n}^-\}, \\ \{\xi_{i,m+1}^+, \xi_{j,n}^+\} - \{\xi_{i,m}^+, \xi_{j,n+1}^+\} &= \frac{1}{2} a_{ij} [\xi_{i,m}^+, \xi_{j,n}^+], \\ \{\xi_{i,m+1}^-, \xi_{j,n}^-\} - \{\xi_{i,m}^-, \xi_{j,n+1}^-\} &= -\frac{1}{2} a_{ij} [\xi_{i,m}^-, \xi_{j,n}^-], \end{aligned} \quad (3.84)$$

$$\begin{aligned} i \neq j, \quad n_{ij} &= 1 + |a_{ij}|, \quad \text{Sym}_{\{k\}}[\xi_{i,k_1}^+, [\xi_{i,k_2}^+, \dots \{\xi_{i,k_{n_{ij}}}^+, \xi_{j,l}^+\} \dots]] = 0, \\ i \neq j, \quad n_{ij} &= 1 + |a_{ij}|, \quad \text{Sym}_{\{k\}}[\xi_{i,k_1}^-, [\xi_{i,k_2}^-, \dots \{\xi_{i,k_{n_{ij}}}^-, \xi_{j,l}^-\} \dots]] = 0, \\ \text{except for } \{\xi_{2,n}^+, \xi_{3,m}^+\} &= \mathbb{C}_{n+m}, \quad \{\xi_{2,n}^-, \xi_{3,m}^-\} = \mathbb{C}_{n+m}^\dagger. \end{aligned} \quad (3.85)$$

In the last relations \mathbb{C}_{n+m} and \mathbb{C}_{n+m}^\dagger are central elements. The exact relation in terms of Drinfeld I operators will be discussed below. The last equation differs from the standard relations due to the central charges and is reminiscent of last Serre relation in the Chevalley-Serre basis of the underlying algebra (3.82). We call the index m of the generators in this realization the level.

We now construct the isomorphism (Drinfeld's map) [114] between the first and the second realization as follows

$$\begin{aligned} \kappa_{i,0} &= \mathfrak{H}_i, \quad \xi_{i,0}^+ = \mathfrak{E}_i^+, \quad \xi_{i,0}^- = \mathfrak{E}_i^-, \\ \kappa_{i,1} &= \hat{\mathfrak{H}}_i - v_i, \quad \xi_{i,1}^+ = \hat{\mathfrak{E}}_i^+ - w_i, \quad \xi_{i,1}^- = \hat{\mathfrak{E}}_i^- - z_i, \end{aligned} \quad (3.86)$$

the special elements are given by

$$\begin{aligned}
v_1 &= -\frac{1}{2}\kappa_{1,0}^2 + \frac{1}{4}\mathbb{R}_3^4\mathbb{R}_4^3 + \frac{1}{4}\mathbb{R}_4^3\mathbb{R}_3^4 + \frac{3}{4}\mathbb{L}_1^2\mathbb{L}_2^1 - \frac{1}{4}\mathbb{L}_2^1\mathbb{L}_1^2 + \\
&\quad -\frac{1}{4}\mathbb{Q}_3^2\mathbb{Q}_2^{\dagger 3} - \frac{1}{4}\mathbb{Q}_4^1\mathbb{Q}_1^{\dagger 4} - \frac{3}{4}\mathbb{Q}_1^{\dagger 4}\mathbb{Q}_4^1 + \frac{1}{4}\mathbb{Q}_2^{\dagger 3}\mathbb{Q}_3^2 + \frac{1}{2}\mathbb{C}\mathbb{C}^\dagger, \\
v_2 &= -\frac{1}{2}\left[\kappa_{2,0}^2 + 2\mathbb{R}_3^4\mathbb{R}_4^3 - \mathbb{R}_4^3\mathbb{R}_3^4 - \mathbb{L}_2^1\mathbb{L}_1^2 - 2\mathbb{Q}_3^1\mathbb{Q}_1^{\dagger 3} - \mathbb{Q}_1^{\dagger 3}\mathbb{Q}_3^1 + \mathbb{Q}_2^{\dagger 4}\mathbb{Q}_4^2 + \mathbb{C}\mathbb{C}^\dagger\right], \\
v_3 &= -\frac{1}{2}\left[\kappa_{3,0}^2 - \mathbb{R}_3^4\mathbb{R}_4^3 + \mathbb{L}_1^2\mathbb{L}_2^1 - \mathbb{Q}_1^{\dagger 3}\mathbb{Q}_3^1 - \mathbb{Q}_2^{\dagger 4}\mathbb{Q}_4^2 + \mathbb{C}\mathbb{C}^\dagger\right], \\
w_1 &= -\frac{1}{4}\left[\xi_{1,0}^+\kappa_{1,0} + \kappa_{1,0}\xi_{1,0}^+ - 3\mathbb{Q}_1^{\dagger 4}\mathbb{L}_2^1 + \mathbb{L}_2^1\mathbb{Q}_1^{\dagger 4} - \mathbb{Q}_2^{\dagger 3}\mathbb{R}_3^4 - \mathbb{R}_3^4\mathbb{Q}_2^{\dagger 3} - 2\mathbb{Q}_3^1\mathbb{C}^\dagger\right], \\
w_2 &= \frac{i}{4}\left[i\xi_{2,0}^+\kappa_{2,0} + i\kappa_{2,0}\xi_{2,0}^+ + 3\mathbb{Q}_3^1\mathbb{R}_4^3 - \mathbb{L}_2^1\mathbb{Q}_4^2 - \mathbb{Q}_4^2\mathbb{L}_2^1 - \mathbb{R}_4^3\mathbb{Q}_3^1 - 2\mathbb{Q}_2^{\dagger 3}\mathbb{C}\right], \\
w_3 &= \frac{i}{4}\left[i\xi_{3,0}^+\kappa_{3,0} + i\kappa_{3,0}\xi_{3,0}^+ - \mathbb{Q}_3^1\mathbb{L}_1^2 + 3\mathbb{L}_1^2\mathbb{Q}_3^1 - \mathbb{Q}_4^2\mathbb{R}_3^4 - \mathbb{R}_3^4\mathbb{Q}_4^2 - 2\mathbb{Q}_1^{\dagger 4}\mathbb{C}\right], \\
z_1 &= -\frac{1}{4}\left[\xi_{1,0}^-\kappa_{1,0} + \kappa_{1,0}\xi_{1,0}^- + \mathbb{Q}_4^1\mathbb{L}_1^2 - 3\mathbb{L}_1^2\mathbb{Q}_4^1 - \mathbb{Q}_3^2\mathbb{R}_4^3 - \mathbb{R}_4^3\mathbb{Q}_3^2 - 2\mathbb{Q}_1^{\dagger 3}\mathbb{C}\right], \\
z_2 &= \frac{i}{4}\left[i\xi_{2,0}^-\kappa_{2,0} + i\kappa_{2,0}\xi_{2,0}^- - \mathbb{Q}_1^{\dagger 3}\mathbb{R}_4^3 + 3\mathbb{R}_4^3\mathbb{Q}_1^{\dagger 3} - \mathbb{Q}_2^{\dagger 4}\mathbb{L}_1^2 - \mathbb{L}_1^2\mathbb{Q}_2^{\dagger 4} - 2\mathbb{Q}_3^2\mathbb{C}^\dagger\right], \\
z_3 &= \frac{i}{4}\left[i\xi_{3,0}^-\kappa_{3,0} + i\kappa_{3,0}\xi_{3,0}^- - \mathbb{Q}_2^{\dagger 4}\mathbb{R}_4^3 - \mathbb{R}_4^3\mathbb{Q}_2^{\dagger 4} + 3\mathbb{Q}_1^{\dagger 3}\mathbb{L}_2^1 - \mathbb{L}_2^1\mathbb{Q}_1^{\dagger 3} - 2\mathbb{Q}_4^1\mathbb{C}^\dagger\right].
\end{aligned}$$

One can check that the above identifications indeed define the second realization of the Yangian (3.84). We can now also make the relation between $\hat{\mathbb{C}}, \hat{\mathbb{C}}^\dagger$ and $\mathbb{C}_1, \mathbb{C}_1^\dagger$ precise. After explicitly computing the corresponding anti-commutator of supercharges we find

$$\mathbb{C}_1 = \hat{\mathbb{C}} - \frac{\mathbb{H} + 1}{2}\mathbb{C}. \quad (3.87)$$

Or, in the evaluation representation (of the first realization)

$$\mathbb{C}_1 = \left(u - \frac{\mathbb{H} + 1}{2}\right)\mathbb{C}. \quad (3.88)$$

Alternatively, if we assume an evaluation representation for the second realization with parameters ω_i

$$\kappa_{i,n} = \omega_i^n \kappa_{i,0}, \quad \xi_{i,n}^+ = \omega_i^n \xi_{i,0}^+, \quad \xi_{i,n}^- = \omega_i^n \xi_{i,0}^-, \quad (3.89)$$

then one has to explicitly work through the defining relations and see whether this ansatz solves it. For example, this works for the fundamental representation where one finds

$$\omega_1 = u, \quad \omega_2 = \omega_3 = u - \frac{1}{2}\mathbb{H}. \quad (3.90)$$

The fact that the parameters $\omega_{2,3}$ are shifted has to do with the modified Serre relations.

3.7 Long Representations

The representations that describe physical particles for the $\text{AdS}_5 \times \text{S}^5$ superstring are the short (atypical) symmetric representations. However, there is also a big class of $\mathfrak{su}(2|2)$ representations that is more general. These are the typical (or long) representations. A very convenient way to construct such representations is to make use of the $\mathfrak{sl}(2)$ outer automorphism (3.21) to construct them out of long representations of $\mathfrak{gl}(2|2)$ [123].

We can identify the values of the labels which will produce the representations we are particularly interested in. First of all, the fundamental 4-dimensional short representation [38] corresponds to $j_1 = \frac{1}{2}, j_2 = 0$ (or, equivalently, $j_1 = 0, j_2 = \frac{1}{2}$) and $q = \frac{1}{2}$ ($q = -\frac{1}{2}$). More generally, the bound state (symmetric short) representations [52–55, 84, 129] are given by $j_2 = 0, q = j_1$, with $j_1 = \frac{1}{2}, 1, \dots$ and bound state number $\ell \equiv s = 2j_1$. In addition, there are the antisymmetric short representations given by $j_1 = 0, q = 1 + j_2$, with $j_2 = \frac{1}{2}, 1, \dots$ and the bound state number $M \equiv a = 2j_2$. Both symmetric and antisymmetric representations have dimension 4ℓ . We see that symmetric and antisymmetric representations are associated with the different shortening conditions $\pm q = j_1 - j_2$ and $\pm q = 1 + j_1 + j_2$.

Second, we consider the simplest long representation of dimension 16. In terms of the $\mathfrak{gl}(2|2)$ labels introduced above, this is the 16-dimensional long representation characterized by $j_1 = j_2 = 0$, and arbitrary q . It is instructive to see how it branches under the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ algebra. We denote as $[l_1, l_2]$ the subset of states which furnish a representation of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ with angular momentum l_1 w.r.t the first $\mathfrak{su}(2)$, and l_2 w.r.t the second $\mathfrak{su}(2)$, respectively. The branching rule is

$$(2, 2) \rightarrow 2 \times [0, 0] \oplus 2 \times \left[\frac{1}{2}, \frac{1}{2}\right] \oplus [1, 0] \oplus [0, 1]. \quad (3.91)$$

One can straightforwardly verify that the total dimension adds up to 16, since $[l_1, l_2]$ has dimension $(2l_1 + 1)(2l_2 + 1)$.

We have explicitly constructed the oscillator representation by using the formulas of [123], and derived from it the 16×16 matrix realization of the algebra generators. We have done this before acting with the outer automorphism, in such a way that the subsequent $\mathfrak{sl}(2)$ rotation provides an explicit matrix representation of centrally-extended $\mathfrak{su}(2|2)$. We report this explicit realization in appendix B. In particular, from the explicit matrix realization one obtains the following values of the central charges:

$$\mathbb{H} = 2q(ad + bc) \mathbb{1}, \quad \mathbb{C} = 2q ab \mathbb{1}, \quad \mathbb{C}^\dagger = 2q cd \mathbb{1}, \quad (3.92)$$

($\mathbb{1}$ is the 16-dimensional identity matrix), satisfying the condition

$$\frac{\mathbb{H}^2}{4} - \mathbb{C}\mathbb{C}^\dagger = q^2 \mathbb{1}. \quad (3.93)$$

When $q^2 = 1$, this becomes a shortening condition. In fact, for $q = 1$ the 16-dimensional representation becomes reducible but indecomposable. Formula (3.93) above, however, tells us that we can conveniently think of q as a *generalized* bound state number, since for short representations $2q$ would be replaced by the bound state number ℓ in the analogous formula for the central charges. This is particularly useful, since it allows us to parameterize the labels a, b, c, d in terms of the familiar bound state variables x^\pm , just replacing the bound state number ℓ by $2q$. The explicit parameterization is given by

$$\begin{aligned} a &= \sqrt{\frac{g}{4q}} \eta, & b &= -\sqrt{\frac{g}{4q}} \frac{i}{\eta} \left(1 - \frac{x^+}{x^-}\right), \\ c &= -\sqrt{\frac{g}{4q}} \frac{\eta}{x^+}, & d &= \sqrt{\frac{g}{4q}} \frac{x^+}{i\eta} \left(1 - \frac{x^-}{x^+}\right), \end{aligned} \quad (3.94)$$

where

$$\eta = e^{\frac{ip}{4}} \sqrt{i(x^- - x^+)} \quad (3.95)$$

and

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{4iq}{g}. \quad (3.96)$$

As in the case of short representations, there exist a uniformizing torus with variable z and periods depending on q [41]. The choice (3.95) for η is historically preferred in the string theory analysis [40, 56, 84, 88], and will again ensure a symmetric S-matrix.

Appendix A: Exceptional Lie algebra

\mathfrak{h} can be obtained via a limiting procedure from the exceptional Lie algebra $D(2, 1; \varepsilon)$ [38, 127]. The advantage is that $D(2, 1; \varepsilon)$ has a non-degenerate Killing form which allows for a standard derivation of the Yangian coproducts. In this appendix we will give the definitions of $D(2, 1; \varepsilon)$, compute its inner product and use it to derive the coproduct of the first Yangian generators. For more details on this exceptional algebra we refer to [130, 131].

The algebra $D(2, 1; \varepsilon)$ consists of three copies of $\mathfrak{su}(2)$

$$\begin{aligned} [\mathbb{L}_b^a, \mathbb{L}_d^c] &= \delta_b^c \mathbb{L}_d^a - \delta_d^a \mathbb{L}_b^c, & [\mathbb{R}_\beta^\alpha, \mathbb{R}_\delta^\gamma] &= \delta_\beta^\gamma \mathbb{R}_\delta^\alpha - \delta_\delta^\alpha \mathbb{R}_\beta^\gamma, \\ [\mathbb{C}_b^a, \mathbb{C}_d^c] &= \delta_b^c \mathbb{C}_d^a - \delta_d^a \mathbb{C}_b^c, \end{aligned} \quad (3.97)$$

and eight supersymmetry generators $\mathbb{F}^{a\alpha a}$ that transform in the fundamental representation of each $\mathfrak{su}(2)$

$$\begin{aligned} [\mathbb{L}_b^a, \mathbb{F}^{c\gamma c}] &= \delta_b^c \mathbb{F}^{a\gamma c} - \frac{1}{2} \delta_b^a \mathbb{F}^{c\gamma c}, & [\mathbb{R}_\beta^\alpha, \mathbb{F}^{c\gamma c}] &= \delta_\beta^\gamma \mathbb{F}^{c\alpha c} - \frac{1}{2} \delta_\beta^\alpha \mathbb{F}^{c\gamma c}, \\ [\mathbb{C}_b^a, \mathbb{F}^{c\gamma c}] &= \delta_b^c \mathbb{F}^{c\gamma a} - \frac{1}{2} \delta_b^a \mathbb{F}^{c\gamma c}. \end{aligned} \quad (3.98)$$

Finally the anti-commutator between the fermionic generators is

$$\{\mathbb{F}^{a\alpha a}, \mathbb{F}^{b\beta b}\} = \sigma_1 \epsilon^{ak} \epsilon^{\alpha\beta} \epsilon^{ab} \mathbb{L}_k^a + \sigma_2 \epsilon^{\beta\kappa} \epsilon^{ab} \epsilon^{ab} \mathbb{R}_\kappa^\alpha + \sigma_3 \epsilon^{ab} \epsilon^{\alpha\beta} \epsilon^{ab} \mathbb{C}_t^a, \quad (3.99)$$

with $\sigma_1 + \sigma_2 + \sigma_3 = 0$. The algebra is invariant under rescaling of the supersymmetry generators and hence the only independent parameter in the algebra is $\varepsilon = -\sigma_3/\sigma_1$. We make the dependence on ε explicit by setting

$$\sigma_1 = -1, \quad \sigma_2 = 1 - \varepsilon, \quad \sigma_3 = \varepsilon. \quad (3.100)$$

To obtain \mathfrak{h} one needs to identify

$$(\mathbb{C})_b^a = \frac{1}{\varepsilon} \begin{pmatrix} \frac{\mathbb{H}}{2} & \mathbb{C}^\dagger \\ -\mathbb{C} & -\frac{\mathbb{H}}{2} \end{pmatrix}, \quad (\mathbb{F}^{a\alpha})^a = \begin{pmatrix} \epsilon^{ak} \mathbb{Q}_k^{\dagger\alpha} \\ \epsilon^{\alpha\kappa} \mathbb{Q}_\kappa^a \end{pmatrix}. \quad (3.101)$$

The above identifications have to be understood in the sense that the $D(2, 1; \varepsilon)$ generators reduce to the $\mathfrak{su}(2|2)$ ones in the limit $\varepsilon \rightarrow 0$, e.g. $\mathbb{C}_1^1 = \frac{\mathbb{H}}{2\varepsilon} + \mathcal{O}(1)$. It is now easily seen that in the limit $\varepsilon \rightarrow 0$ the commutation relations (3.97), (3.98), (3.99) reduce to (3.1), e.g. we see that

$$\{\mathbb{Q}_2^{\dagger 3}, \mathbb{Q}_3^2\} = -\{\mathbb{F}^{111}, \mathbb{F}^{222}\} = -\mathbb{L}_1^1 + (1 - \varepsilon)\mathbb{R}_4^4 + \frac{1}{2}\mathbb{H}. \quad (3.102)$$

It is also readily checked that the elements \mathbb{C}_b^a become indeed central.

Normally in superalgebras one lowers indices by making use of the Killing form K^{AB} . The Killing form for superalgebras is defined as

$$K^{AB} = \text{str}(\text{ad}(\mathbb{J}^A)\text{ad}(\mathbb{J}^B)). \quad (3.103)$$

Computing this from the commutation relations is straightforward and we find

$$K^{AB} = (-1)^{d(D)} f_D^{AC} f_C^{BD} = 0, \quad (3.104)$$

where $d(A) = 0, 1$ for bosonic and fermionic generators respectively. Nevertheless, $D(2, 1; \varepsilon)$ admits an invariant, non-degenerate inner product. An inner product $\tilde{K}(\mathbb{J}^A, \mathbb{J}^B) \equiv \tilde{K}^{AB}$ on a Lie superalgebra needs to satisfy [132]

$$\tilde{K}(\mathbb{J}^A, \mathbb{J}^B) = 0 \text{ if } d(A) \neq d(B) \quad (3.105)$$

$$\tilde{K}(\mathbb{J}^A, \mathbb{J}^B) = (-1)^{d(A)d(B)} \tilde{K}(\mathbb{J}^B, \mathbb{J}^A) \quad (3.106)$$

$$\tilde{K}(\mathbb{J}^A, [\mathbb{J}^B, \mathbb{J}^C]) = \tilde{K}([\mathbb{J}^A, \mathbb{J}^B], \mathbb{J}^C). \quad (3.107)$$

In terms of components and structure constants, the last equation becomes $\tilde{K}^{AD} f_D^{BC} = \tilde{K}^{DC} f_D^{AB}$ for all A, B, C . Solving it gives an unique solution (up to an overall scalar) for \tilde{K}^{AB} . Let us

enumerate the algebra generators as

$$J[1] = \mathbb{L}_1^1 \quad J[2] = \mathbb{L}_1^2 \quad J[3] = \mathbb{L}_2^1 \quad (3.108)$$

$$J[4] = \mathbb{R}_1^1 \quad J[5] = \mathbb{R}_1^2 \quad J[6] = \mathbb{R}_2^1 \quad (3.109)$$

$$J[7] = \mathbb{C}_1^1 \quad J[8] = \mathbb{C}_1^2 \quad J[9] = \mathbb{C}_2^1 \quad (3.110)$$

and supersymmetry generators

$$J[10] = \mathbb{F}^{111} \quad J[11] = \mathbb{F}^{112} \quad J[12] = \mathbb{F}^{121} \quad J[13] = \mathbb{F}^{211} \quad (3.111)$$

$$J[14] = \mathbb{F}^{122} \quad J[15] = \mathbb{F}^{212} \quad J[16] = \mathbb{F}^{221} \quad J[17] = \mathbb{F}^{222}. \quad (3.112)$$

Then the elements of the inner product can be conveniently encoded in a matrix and are given by

$$\tilde{K}^{AB} = \begin{pmatrix} \frac{1}{\sigma_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{\sigma_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{\sigma_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sigma_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{\sigma_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{\sigma_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sigma_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{\sigma_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{\sigma_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.113)$$

In order to compute the coproducts of the Yangian generators we need to lower indices. This is done by the contravariant form \tilde{K}_{AB} . In matrix form it becomes

$$\tilde{K}_{AB} = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sigma_1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sigma_1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sigma_2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sigma_2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sigma_3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sigma_3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.114)$$

This form defines the two-site Casimir

$$T_{12} = \tilde{K}_{AB} \mathbb{J}^A \otimes \mathbb{J}^B. \quad (3.115)$$

The operator T_{12} can for example be used to compute the classical r -matrix of this algebra. Computing the coproduct of the Yangian generator is now straightforward from (2.33) by $f_{BC}^A = f_C^{AB} \tilde{K}_{DB}$ and gives (after including braiding factors)

$$\Delta(\hat{\mathbb{L}}_b^a) = \hat{\mathbb{L}}_b^a \otimes 1 + 1 \otimes \hat{\mathbb{L}}_b^a + \frac{1}{2} \left[-(\mathbb{L}_b^c \otimes \mathbb{L}_c^a - \mathbb{L}_c^a \otimes \mathbb{L}_b^c) + \right. \quad (3.116)$$

$$\left. -\mathbb{Q}_b^{\dagger\gamma} \otimes \mathbb{U}^{-1} \mathbb{Q}_\gamma^a - \mathbb{Q}_\gamma^a \otimes \mathbb{U} \mathbb{Q}_b^{\dagger\gamma} + \frac{\delta_b^a}{2} (\mathbb{Q}_c^{\dagger\gamma} \otimes \mathbb{U}^{-1} \mathbb{Q}_\gamma^c + \mathbb{Q}_\gamma^c \otimes \mathbb{U} \mathbb{Q}_c^{\dagger\gamma}) \right]$$

$$\Delta(\hat{\mathbb{R}}_\beta^\alpha) = \hat{\mathbb{R}}_\beta^\alpha \otimes 1 + 1 \otimes \hat{\mathbb{R}}_\beta^\alpha + \frac{1}{2} \left[(1 + \varepsilon) (\mathbb{R}_\beta^\gamma \otimes \mathbb{R}_\gamma^\alpha - \mathbb{R}_\gamma^\alpha \otimes \mathbb{R}_\beta^\gamma) + \right. \quad (3.117)$$

$$\left. + \mathbb{Q}_c^{\dagger\alpha} \otimes \mathbb{U}^{-1} \mathbb{Q}_\beta^c + \mathbb{Q}_\beta^c \otimes \mathbb{U} \mathbb{Q}_c^{\dagger\alpha} - \frac{\delta_\beta^\alpha}{2} (\mathbb{Q}_c^{\dagger\gamma} \otimes \mathbb{U}^{-1} \mathbb{Q}_\gamma^c + \mathbb{Q}_\gamma^c \otimes \mathbb{U} \mathbb{Q}_c^{\dagger\gamma}) \right]$$

$$\Delta(\hat{\mathbb{Q}}_\alpha^a) = \hat{\mathbb{Q}}_\alpha^a \otimes \mathbb{U} + 1 \otimes \hat{\mathbb{Q}}_\alpha^a + \frac{1}{2} \left[(1 - \varepsilon) (\mathbb{Q}_\gamma^a \otimes \mathbb{U} \mathbb{R}_\alpha^\gamma - \mathbb{R}_\alpha^\gamma \otimes \mathbb{Q}_\gamma^a) + \right. \quad (3.118)$$

$$\left. -\mathbb{L}_{1;c}^a \otimes \mathbb{Q}_\alpha^c + \mathbb{Q}_\alpha^c \otimes \mathbb{U} \mathbb{L}_c^a - \frac{1}{2} \mathbb{H}_1 \otimes \mathbb{Q}_\alpha^a + \frac{1}{2} \mathbb{Q}_\alpha^a \otimes \mathbb{U} \mathbb{H} + \right. \\ \left. + \epsilon_{\alpha\gamma} \epsilon^{ac} \mathbb{C} \otimes \mathbb{U}^2 \mathbb{Q}_c^{\dagger\gamma} - \epsilon_{\alpha\gamma} \epsilon^{ac} \mathbb{Q}_c^{\dagger\gamma} \otimes \mathbb{U} \mathbb{C} \right],$$

$$\Delta(\hat{\mathbb{Q}}_a^{\dagger\alpha}) = \hat{\mathbb{Q}}_a^{\dagger\alpha} \otimes \mathbb{U}^{-1} + 1 \otimes \hat{\mathbb{Q}}_a^{\dagger\alpha} + \frac{1}{2} \left[\mathbb{L}_a^c \otimes \mathbb{Q}_a^{\dagger\alpha} - \mathbb{Q}_c^{\dagger\alpha} \otimes \mathbb{U}^{-1} \mathbb{L}_a^c + \right. \quad (3.119)$$

$$\left. + (1 - \varepsilon) (\mathbb{R}_\gamma^\alpha \otimes \mathbb{Q}_a^{\dagger\gamma} - \mathbb{Q}_a^{\dagger\gamma} \otimes \mathbb{U}^{-1} \mathbb{R}_\gamma^\alpha) + \frac{1}{2} \mathbb{H} \otimes \mathbb{Q}_a^{\dagger\alpha} - \frac{1}{2} \mathbb{Q}_a^{\dagger\alpha} \otimes \mathbb{U}^{-1} \mathbb{H} + \right. \\ \left. - \epsilon_{ac} \epsilon^{\alpha\gamma} \mathbb{C}^\dagger \otimes \mathbb{U}^{-2} \mathbb{Q}_\gamma^c + \epsilon_{ac} \epsilon^{\alpha\gamma} \mathbb{Q}_\gamma^c \otimes \mathbb{U} \mathbb{C}^\dagger \right].$$

The coproducts of the central charges become

$$\Delta(\hat{\mathbb{H}}) = \hat{\mathbb{H}} \otimes 1 + 1 \otimes \hat{\mathbb{H}} + \mathbb{C} \otimes \mathbb{U}^2 \mathbb{C}^\dagger - \mathbb{C}^\dagger \otimes \mathbb{U}^{-2} \mathbb{C} + \quad (3.120)$$

$$+ \frac{\varepsilon}{2} (\mathbb{Q}_\alpha^a \otimes \mathbb{U} \mathbb{Q}_a^{\dagger\alpha} + \mathbb{Q}_a^{\dagger\alpha} \otimes \mathbb{U}^{-1} \mathbb{Q}_\alpha^a),$$

$$\Delta(\hat{\mathbb{C}}) = \hat{\mathbb{C}} \otimes \mathbb{U}^2 + 1 \otimes \hat{\mathbb{C}} + \frac{1}{2} [\mathbb{H} \otimes \mathbb{C} - \mathbb{C} \otimes \mathbb{U}^2 \mathbb{H} + \quad (3.121)$$

$$+ \frac{\varepsilon}{2} (\epsilon_{ab} \epsilon^{\alpha\beta} \mathbb{Q}_\alpha^a \otimes \mathbb{U} \mathbb{Q}_\beta^b)],$$

$$\Delta(\hat{\mathbb{C}}^\dagger) = \hat{\mathbb{C}}^\dagger \otimes \mathbb{U}^{-2} + 1 \otimes \hat{\mathbb{C}}^\dagger - \frac{1}{2} [\mathbb{H} \otimes \mathbb{C}^\dagger - \mathbb{C}^\dagger \otimes \mathbb{U}^{-2} \mathbb{H} + \quad (3.122)$$

$$+ \frac{\varepsilon}{2} (\epsilon^{ab} \epsilon_{\alpha\beta} \mathbb{Q}_a^{\dagger\alpha} \otimes \mathbb{U}^{-1} \mathbb{Q}_b^{\dagger\beta})].$$

Upon taking the limit $\varepsilon \rightarrow 0$ one reproduces the $\mathfrak{su}(2|2)$ Yangian coproducts presented above.

We list in this appendix the generators of centrally extended $\mathfrak{su}(2|2)$ in the long representation. We only report explicitly the simple roots for a distinguished Dynkin diagram, the remainder of the algebra being generated via commutation relations. We present the roots in a unitary representation. To achieve this, we perform a similarity transformation on the generators constructed directly from the oscillator basis of [123], in order to obtain hermitean matrices. First, the bosonic $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ roots are given by

[illegible]

[illegible]
$$\mathbb{Q}_3^1 = \begin{pmatrix} 0 & 0 & 0 & -b\sqrt{q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & bq_- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a\sqrt{q} & 0 & 0 & 0 & 0 & 0 & \frac{bq_-}{\sqrt{2}} & 0 & 0 & \frac{bq_+}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & bq_+ & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{aq_-}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{bq_+}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & aq_- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -bq_+ & 0 \\ 0 & aq_+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & bq_- & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{aq_+}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{bq_-}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -aq_+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{aq_+}{\sqrt{2}} & 0 & 0 & \frac{aq_-}{\sqrt{2}} & 0 & 0 & 0 & 0 & -b\sqrt{q} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & aq_- & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a\sqrt{q} & 0 & 0 \end{pmatrix}, \quad (3.125)$$

and

$$\mathbb{Q}_2^{\dagger 4} = \begin{pmatrix} 0 & 0 & 0 & -d\sqrt{q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & dq_- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c\sqrt{q} & 0 & 0 & 0 & 0 & 0 & \frac{dq_-}{\sqrt{2}} & 0 & \frac{dq_+}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & dq_+ & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{cq_-}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{dq_+}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & cq_- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -dq_+ & 0 \\ 0 & cq_+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & dq_- & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{cq_+}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{dq_-}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -cq_+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{cq_+}{\sqrt{2}} & 0 & \frac{cq_-}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -d\sqrt{q} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & cq_- & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c\sqrt{q} & 0 & 0 \end{pmatrix}. \quad (3.126)$$

For completeness let us also explicitly give the similarity transformation that relates the unitary representation to the one from [123]

$$\mathcal{V} = \text{diag}(\sqrt{q^3 - q}, q_+q_-, q_+q_-, q_+q_-, q_+q_-, 2q_+, \sqrt{2}q_+, 2q_+, 2q_-, \sqrt{2}q_-, 2q_-, 1, 1, 1, 1, \frac{1}{\sqrt{q}}) \quad (3.127)$$

We notice that this transformation is singular for $q^2 = 1$, where the representation becomes reducible but indecomposable.

Chapter 4

Bound State S-Matrices

The light cone gauge-fixed Hamiltonian of the $\text{AdS}_5 \times \text{S}^5$ superstring admits two copies of the centrally extended $\mathfrak{su}(2|2)$ (\mathfrak{h}) as a symmetry algebra [39]. World-sheet excitations transform in the fundamental representation of this algebra, hence their scattering data is encoded in the S-matrix of \mathfrak{h} in this representation. The same algebra emerges in $\mathcal{N} = 4$ SYM, where it appears as the algebra governing a spin chain whose energies encode the anomalous dimensions of large operators [38].

Now that we have discussed the symmetry algebra in detail and also studied its Yangian in the previous chapter, we can put it to use in the computation of S-matrices. First we will discuss the fundamental S-matrix. We will give its explicit form and indicate the pole structure that gives rise to bound states. We will also discuss some of its properties including Yangian symmetry. By assuming Yangian symmetry we will be able to give a complete derivation of the S-matrix that describes scattering between arbitrary bound states, reproducing all data known so far.

4.1 The Fundamental S-matrix

The S-matrix describing the scattering of fundamental excitations is given by the S-matrix of \mathfrak{h} seen as a Hopf algebra. Since we explicitly know the fundamental representation it is straightforward to compute the 16×16 dimensional matrix that intertwines the coproduct and the opposite coproduct. Consider the fundamental representation $V^F(p)$ depending on the momentum p and coupling constant g . The S-matrix $\mathbb{S}^F : V^F(p_1) \otimes V^F(p_2) \rightarrow V^F(p_1) \otimes V^F(p_2)$

is given by

$$\mathbb{S}^F = \begin{pmatrix} \begin{array}{cccc|cccc|cccc|cccc} a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1+a_2 & 0 & 0 & -a_2 & 0 & 0 & 0 & 0 & 0 & 0 & -a_7 & 0 & 0 \\ 0 & 0 & a_5 & 0 & 0 & 0 & 0 & 0 & a_9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_9 & 0 \\ \hline 0 & -a_2 & 0 & 0 & a_1+a_2 & 0 & 0 & 0 & 0 & 0 & 0 & a_7 & 0 & -a_7 \\ 0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_5 & 0 & 0 & a_9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_5 & 0 & 0 & 0 & 0 & a_9 & 0 \\ \hline 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & a_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 \\ 0 & -a_8 & 0 & 0 & a_8 & 0 & 0 & 0 & 0 & 0 & 0 & a_3+a_4 & 0 & 0 \\ \hline 0 & 0 & 0 & a_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{10} & 0 & 0 & 0 & 0 & a_6 & 0 \\ 0 & a_8 & 0 & 0 & -a_8 & 0 & 0 & 0 & 0 & 0 & 0 & -a_4 & 0 & a_3+a_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 \end{array} \end{pmatrix} \quad (4.1)$$

with coefficients

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 2 \frac{(x_2^+ - x_1^+)(x_1^- x_2^+ - 1)x_2^-}{(x_2^- - x_1^+)(x_1^- x_2^- - 1)x_2^+} - 1 \\ a_3 &= \frac{x_2^+ - x_1^-}{x_2^- - x_1^+} \sqrt{\frac{x_1^+}{x_1^-}} \sqrt{\frac{x_2^-}{x_2^+}} \\ a_4 &= \frac{x_1^- - x_2^+}{x_2^- - x_1^+} \sqrt{\frac{x_1^+}{x_1^-}} \sqrt{\frac{x_2^-}{x_2^+}} - 2 \frac{(x_2^- x_1^+ - 1)(x_1^+ - x_2^+)x_1^-}{(x_1^- x_2^- - 1)(x_2^- - x_1^+)x_1^+} \sqrt{\frac{x_1^+}{x_1^-}} \sqrt{\frac{x_2^-}{x_2^+}} \\ a_5 &= \frac{x_1^- - x_2^-}{x_1^+ - x_2^-} \sqrt{\frac{x_1^+}{x_1^-}} \\ a_6 &= \frac{x_2^+ - x_1^+}{x_2^- - x_1^+} \sqrt{\frac{x_2^-}{x_2^+}} \\ a_7 &= -\frac{i(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_1^- x_2^- - 1)(x_2^- - x_1^+)\eta(p_1)\eta(p_2)} \sqrt{\frac{x_2^-}{x_2^+}} \\ a_8 &= \frac{i(x_1^+ - x_2^+)\eta(p_1)\eta(p_2)x_1^- x_2^-}{(x_1^- x_2^- - 1)(x_2^- - x_1^+)x_1^+ x_2^+} \sqrt{\frac{x_1^+}{x_1^-}} \\ a_9 &= \frac{x_1^- - x_1^+}{x_2^- - x_1^+} \sqrt{\frac{x_1^+}{x_1^-}} \sqrt{\frac{x_2^-}{x_2^+}} \frac{\eta(p_2)}{\eta(p_1)} \\ a_{10} &= \frac{x_2^- - x_2^+}{x_2^- - x_1^+} \frac{\eta(p_1)}{\eta(p_2)}, \end{aligned}$$

where the parameters x_i^\pm are related to the momentum p_i via (3.12). We have normalized the S-matrix in such a way that $a_1 = 1$. The S-matrix satisfies the identities common to S-matrices to two-dimensional integrable systems

Unitarity: $\mathbb{S}_{12}^F(z_1, z_2) \mathbb{S}_{21}^F(z_2, z_1) = \mathbb{1}.$

Hermiticity: $\mathbb{S}_{12}^F(z_1, z_2) \mathbb{S}_{12}^F(z_1^*, z_2^*)^\dagger = \mathbb{1}.$

CPT Invariance: $\mathbb{S}_{12}^F = (\mathbb{S}_{12}^F)^t.$

Yang-Baxter: $\mathbb{S}_{12}^F \mathbb{S}_{13}^F \mathbb{S}_{23}^F = \mathbb{S}_{23}^F \mathbb{S}_{13}^F \mathbb{S}_{12}^F.$

The full S-matrix includes an overall scalar factor $S_0(p_1, p_2)$, which is not fixed by the invariance condition (2.28). However crossing symmetry puts restrictions on it [41]. The problem of finding the appropriate overall scalar factor which reproduces the correct anomalous dimensions has been extensively studied in the literature [42, 43, 133–135]. An exact conjecture has been put forward in [44]

$$S_F(p_1, p_2) = \left(\frac{x_1^-}{x_1^+} \right)^{\frac{1}{2}} \left(\frac{x_2^+}{x_2^-} \right)^{\frac{1}{2}} \sigma(x_1, x_2) \sqrt{G(2)}. \quad (4.2)$$

where

$$G(n) = \frac{u_1 - u_2 + \frac{n}{2}}{u_1 - u_2 - \frac{n}{2}}, \quad \sigma(p_1, p_2) = e^{\frac{i}{2}\theta(p_1, p_2)}. \quad (4.3)$$

The function $\theta(p_1, p_2)$ is called the dressing phase and it is defined in terms of conserved charges [42]

$$q_n(x_i) = \frac{i}{n-1} \left(\frac{1}{(x_i^+)^{n-1}} - \frac{1}{(x_i^-)^{n-1}} \right). \quad (4.4)$$

as follows:

$$\theta(p_1, p_2) = \sum_{r=2}^{\infty} \sum_{n=0}^{\infty} c_{r,r+1+2n} (q_r(p_1) q_{r+1+2n}(p_2) - q_r(p_2) q_{r+1+2n}(p_1)), \quad (4.5)$$

for some coefficients c . This solution satisfies crossing symmetry and agrees with all data known so far from string and gauge perturbation theory.

As is not uncommon in integrable field theories, the S-matrix actually respects a bigger symmetry group, namely the Yangian of \mathfrak{h} . It is readily checked that this matrix intertwines the different coproducts of the Yangian (3.67) in the evaluation representation [83].

Finally, the S-matrix has a pole at $x_1^+ = x_2^-$. This indicates the presence of bound states [56]. The S-matrices corresponding to scattering of bound states of up to two fundamental particles have been computed in [84]. It appears [85] that in these cases the requirement of the Yang-Baxter equation is equivalent to the presence of Yangian symmetry (see also [40]). In what follows we will make use of this fact and assume Yangian symmetry to be present for all bound state S-matrices.

4.2 Kinematical Structure of the S-Matrix

It is useful to take a closer look at the kinematical structure of the S-matrix. In particular, we will use \mathfrak{h} invariance to show that the S-matrix is of block diagonal form. The bound state S-matrices should respect the \mathfrak{h} symmetry, which is imposed by requiring invariance under the coproducts of the generators (2.28)

$$\mathbb{S} \Delta(\mathbb{J}^A) = \Delta^{op}(\mathbb{J}^A) \mathbb{S}. \quad (4.6)$$

This formula will be our starting point. We consider for the generators the bound state representations in the superspace formalism discussed in the previous chapter.

4.2.1 Invariant subspaces

Consider two short symmetric solutions with bound state numbers ℓ_1, ℓ_2 respectively (cf. section 3.4). The tensor product of the corresponding bound state representations in superspace [84] is given by:

$$\Phi(w, \theta) \Phi(v, \vartheta), \quad (4.7)$$

where w, θ denote the superspace variables of the first particle and v, ϑ describe the representation of the second particle.

The S-matrix acts on this tensor space and should, according to (4.6), commute in particular with $\Delta \mathbb{L}_1^1 = \Delta^{op} \mathbb{L}_1^1$ and $\Delta \mathbb{R}_3^3 = \Delta^{op} \mathbb{R}_3^3$. From this, it is easily deduced that the numbers

$$\begin{aligned} K^{\text{II}} &\equiv \# \theta_3 + \# \theta_4 + \# \vartheta_3 + \# \vartheta_4 + 2\# w_2 + 2\# v_2, \\ K^{\text{III}} &\equiv \# \theta_3 + \# \vartheta_3 + \# w_2 + \# v_2 \end{aligned} \quad (4.8)$$

are conserved. Here $\# w_2$ means the number of w_2 's, i.e. $\# w_2$ of the state w_2^k is k , etc. More precisely, for any state with bound state number ℓ we have

$$(\ell - \mathbb{L}_1^1) \theta_3^k \theta_4^l w_1^m w_2^n = (2n + k + l) \theta_3^k \theta_4^l w_1^m w_2^n \quad (4.9)$$

$$(\ell - \mathbb{L}_1^1 + \mathbb{R}_3^3) \theta_3^k \theta_4^l w_1^m w_2^n = (2n + 2k) \theta_3^k \theta_4^l w_1^m w_2^n \quad (4.10)$$

Applying these formulas to the coproducts projected into the tensor product of two bound states we obtain the above expressions for $K^{\text{II}}, K^{\text{III}}$.

The variables w_2, v_2 can be interpreted as being a combined state of two fermions of different type [38]. Hence, the number K^{II} corresponds to the total number of fermions, and the number K^{III} counts the number of fermions of type 3. The fact that these numbers are conserved allows us to define subspaces that the S-matrix has to leave invariant. For each of these subspaces we will derive the corresponding S-matrix.

Let us write out the tensor product more explicitly. Since we are considering bound states with bound state number ℓ_1, ℓ_2 we restrict to

$$(w_1^{\ell_1-k} w_2^k + \theta_3 w_1^{\ell_1-k-1} w_2^k + \theta_4 w_1^{\ell_1-k-1} w_2^k + \theta_3 \theta_4 w_1^{\ell_1-k-1} w_2^{k-1}) \times \\ \times (v_1^{\ell_2-l} v_2^l + \vartheta_3 v_1^{\ell_2-l-1} v_2^l + \vartheta_4 v_1^{\ell_2-l-1} v_2^l + \vartheta_3 \vartheta_4 v_1^{\ell_2-l-1} v_2^{l-1}). \quad (4.11)$$

The ranges over which the labels k, l are allowed to vary can be straightforwardly read off for each term. By multiplying everything out, we reproduce the basis vectors that span the tensor product representation of these two bound states. One can compute the quantum numbers $K^{\text{II}}, K^{\text{III}}$ for any of these basis vectors. The results are listed in Table 4.1. When we take a

Space 1	Space 2	K^{II}	K^{III}	N	Case
$\theta_3 w_1^{\ell_1-k-1} w_2^k$	$\vartheta_3 v_1^{\ell_2-l-1} v_2^l$	$2(k+l)+2$	$k+l+2$	$k+l$	Ia
$\theta_4 w_1^{\ell_1-k-1} w_2^k$	$\vartheta_4 v_1^{\ell_2-l-1} v_2^l$	$2(k+l)+2$	$k+l$	$k+l$	Ib
$\theta_3 w_1^{\ell_1-k-1} w_2^k$	$v_2^{\ell_2-l} v_2^l$	$2(k+l)+1$	$k+l+1$	$k+l$	IIa
$w_1^{\ell_1-k} w_2^k$	$\vartheta_3 v_1^{\ell_2-l-1} v_2^l$	$2(k+l)+1$	$k+l+1$	$k+l$	IIa
$\theta_3 w_1^{\ell_1-k-1} w_2^k$	$\vartheta_3 \vartheta_4 v_1^{\ell_2-l-1} v_2^{l-1}$	$2(k+l)+1$	$k+l+1$	$k+l$	IIa
$\theta_3 \theta_4 w_1^{\ell_1-k-1} w_2^{k-1}$	$\vartheta_3 v_1^{\ell_2-l-1} v_2^l$	$2(k+l)+1$	$k+l+1$	$k+l$	IIa
$\theta_4 w_1^{\ell_1-k-1} w_2^k$	$v_2^{\ell_2-l} v_2^l$	$2(k+l)+1$	$k+l$	$k+l$	IIb
$w_1^{\ell_1-k} w_2^k$	$\vartheta_4 v_1^{\ell_2-l-1} v_2^l$	$2(k+l)+1$	$k+l$	$k+l$	IIb
$\theta_4 w_1^{\ell_1-k-1} w_2^k$	$\vartheta_3 \vartheta_4 v_1^{\ell_2-l-1} v_2^{l-1}$	$2(k+l)+1$	$k+l$	$k+l$	IIb
$\theta_3 \theta_4 w_1^{\ell_1-k-1} w_2^{k-1}$	$\vartheta_4 v_1^{\ell_2-l-1} v_2^l$	$2(k+l)+1$	$k+l$	$k+l$	IIb
$w_1^{\ell_1-k} w_2^k$	$v_2^{\ell_2-l} v_2^l$	$2(k+l)$	$k+l$	$k+l$	III
$w_1^{\ell_1-k} w_2^k$	$\vartheta_3 \vartheta_4 v_1^{\ell_2-l-1} v_2^{l-1}$	$2(k+l)$	$k+l$	$k+l$	III
$\theta_3 \theta_4 w_1^{\ell_1-k-1} w_2^{k-1}$	$v_2^{\ell_2-l} v_2^l$	$2(k+l)$	$k+l$	$k+l$	III
$\theta_3 \theta_4 w_1^{\ell_1-k-1} w_2^{k-1}$	$\vartheta_3 \vartheta_4 v_1^{\ell_2-l-1} v_2^{l-1}$	$2(k+l)$	$k+l$	$k+l$	III
$\theta_3 w_1^{\ell_1-k-1} w_2^k$	$\vartheta_4 v_1^{\ell_2-l-1} v_2^l$	$2(k+l+1)$	$k+l+1$	$k+l+1$	III
$\theta_4 w_1^{\ell_1-k-1} w_2^k$	$\vartheta_3 v_1^{\ell_2-l-1} v_2^l$	$2(k+l+1)$	$k+l+1$	$k+l+1$	III

Table 4.1: The $16\ell_1\ell_2$ vectors from the tensor product and their $\mathfrak{su}(2) \times \mathfrak{su}(2)$ quantum numbers.

closer look at the result, we see that ordering the states by the quantum numbers $K^{\text{II}}, K^{\text{III}}$, there are exactly five different types of states:

Case Ia: $K^{\text{II}} = 2N + 2, K^{\text{III}} = N + 2,$

Case Ib: $K^{\text{II}} = 2N + 2, K^{\text{III}} = N,$

Case IIa: $K^{\text{II}} = 2N + 1, K^{\text{III}} = N + 1,$

Case IIb: $K^{\text{II}} = 2N + 1, K^{\text{III}} = N,$

Case III: $K^{\text{II}} = 2N, K^{\text{III}} = N,$

for some integer N . For fixed N , each of these states has different quantum numbers $K^{\text{II}}, K^{\text{III}}$ and hence the states belonging to each of these cases form an invariant subspace under the action of the S-matrix.

Clearly, vectors from Case Ia and Case Ib only differ by the exchange of the fermionic index $3 \leftrightarrow 4$, which is easily realized in terms of the (fermionic) $\mathfrak{su}(2)$ symmetry generators of type \mathbb{R} . Hence, the subspaces spanned by the two types of states are isomorphic, and scatter via the same S-matrix. An analogous relationship connects Case IIa and IIb. Thus, there are only three non-equivalent cases:

Case I: $K^{\text{II}} = 2N + 2, K^{\text{III}} = N + 2,$

Case II: $K^{\text{II}} = 2N + 1, K^{\text{III}} = N + 1,$

Case III: $K^{\text{II}} = 2N, K^{\text{III}} = N.$

For fixed N (i.e. for fixed $K^{\text{II}}, K^{\text{III}}$) we denote the vector spaces spanned by vectors from each of the inequivalent cases by $V_N^{\text{I}}, V_N^{\text{II}}, V_N^{\text{III}}$ respectively. In what follows we will compute the S-matrix for each of these invariant subspaces. For this we will first need to study these invariant spaces in more detail.

4.2.2 Basis and relations

Let us give a complete description of the bases of the invariant subspaces. Later on we will introduce different choices of basis for the different cases, but in this section we will discuss the bases as obtained simply by multiplying out the tensor product as seen from Table 4.1. We will call this type of basis the standard one.

Case I, $K^{\text{II}} = 2N + 2, K^{\text{III}} = N + 2.$

For fixed N , the vector space of states V_N^{I} is $N + 1$ -dimensional. The standard basis for this vector space is

$$|k, l\rangle^{\text{I}} \equiv \underbrace{\theta_3 w_1^{\ell_1 - k - 1} w_2^k}_{\text{Space1}} \underbrace{\vartheta_3 v_1^{\ell_2 - l - 1} v_2^l}_{\text{Space2}}, \quad (4.12)$$

for all $k + l = N$. These indeed give $N + 1$ different vectors.

Case II, $K^{\text{II}} = 2N + 1, K^{\text{III}} = N + 1$.

For fixed N , the dimension of this vector space is $4N + 2$. The standard basis is

$$\begin{aligned}
|k, l\rangle_1^{\text{II}} &\equiv \underbrace{\theta_3 w_1^{\ell_1 - k - 1} w_2^k}_{\text{Space 1}} \underbrace{v_1^{\ell_2 - l} v_2^l}_{\text{Space 2}}, \\
|k, l\rangle_2^{\text{II}} &\equiv \underbrace{w_1^{\ell_1 - k} w_2^k}_{\text{Space 1}} \underbrace{\vartheta_3 v_1^{\ell_2 - l - 1} v_2^l}_{\text{Space 2}}, \\
|k, l\rangle_3^{\text{II}} &\equiv \underbrace{\theta_3 w_1^{\ell_1 - k - 1} w_2^k}_{\text{Space 1}} \underbrace{\vartheta_3 \vartheta_4 v_1^{\ell_2 - l - 1} v_2^{l-1}}_{\text{Space 2}}, \\
|k, l\rangle_4^{\text{II}} &\equiv \underbrace{\theta_3 \theta_4 w_1^{\ell_1 - k - 1} w_2^{k-1}}_{\text{Space 1}} \underbrace{\vartheta_3 v_1^{\ell_2 - l - 1} v_2^l}_{\text{Space 2}},
\end{aligned} \tag{4.13}$$

where $k + l = N$. As a lighter notation, we will from now on, with no risk of confusion, omit indicating “Space 1” and “Space 2” under the curly brackets. The ranges of k, l are clear from the explicit expressions and it is easily seen that we get $4N + 2$ states.

Case III: $K^{\text{II}} = 2N, K^{\text{III}} = N$

For fixed $N = k + l$, the dimension of this vector space is $6N$. The standard basis is

$$\begin{aligned}
|k, l\rangle_1^{\text{III}} &\equiv \underbrace{w_1^{\ell_1 - k} w_2^k}_{\text{Space 1}} \underbrace{v_1^{\ell_2 - l} v_2^l}_{\text{Space 2}}, \\
|k, l\rangle_2^{\text{III}} &\equiv \underbrace{w_1^{\ell_1 - k} w_2^k}_{\text{Space 1}} \underbrace{\vartheta_3 \vartheta_4 v_1^{\ell_2 - l - 1} v_2^{l-1}}_{\text{Space 2}}, \\
|k, l\rangle_3^{\text{III}} &\equiv \underbrace{\theta_3 \theta_4 w_1^{\ell_1 - k - 1} w_2^{k-1}}_{\text{Space 1}} \underbrace{v_1^{\ell_2 - l} v_2^l}_{\text{Space 2}}, \\
|k, l\rangle_4^{\text{III}} &\equiv \underbrace{\theta_3 \theta_4 w_1^{\ell_1 - k - 1} w_2^{k-1}}_{\text{Space 1}} \underbrace{\vartheta_3 \vartheta_4 v_1^{\ell_2 - l - 1} v_2^{l-1}}_{\text{Space 2}}, \\
|k, l\rangle_5^{\text{III}} &\equiv \underbrace{\theta_3 w_1^{\ell_1 - k - 1} w_2^k}_{\text{Space 1}} \underbrace{\vartheta_4 v_1^{\ell_2 - l} v_2^{l-1}}_{\text{Space 2}}, \\
|k, l\rangle_6^{\text{III}} &\equiv \underbrace{\theta_4 w_1^{\ell_1 - k} w_2^{k-1}}_{\text{Space 1}} \underbrace{\vartheta_3 v_1^{\ell_2 - l - 1} v_2^l}_{\text{Space 2}}.
\end{aligned} \tag{4.14}$$

Note that our numbering slightly differs from the one used in Table 4.1, in the sense that $|k, l\rangle_{5,6}^{\text{III}}$ are rescaled for convenience in such a way that they also have $N = k + l$, instead of $k + l + 1$ as in Table 4.1.

It is useful to supply all these spaces with a canonical inner product:

$${}_j^A \langle k, l | m, n \rangle_i^B = \delta_{ij} \delta_{km} \delta_{ln} \delta_{AB}. \tag{4.15}$$

Actually, for the sake of our arguments, orthogonality of these vectors will always be sufficient.

For later convenience we also introduce the vector spaces $V_{k,l}^A = \text{span}\{|k, l\rangle_i^A\}$, for $A = \text{I, II, III}$. These subspaces are generated by the basis elements for fixed k, l and together build up the full invariant subspace

$$V_N^A = \bigoplus_{k+l=N} V_{k,l}^A. \tag{4.16}$$

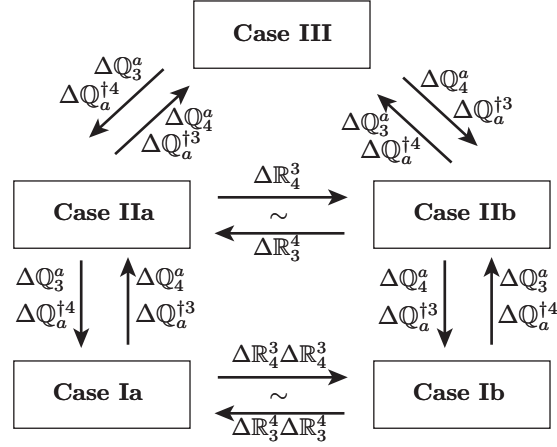


Figure 4.1: Schematic representation of the relations between the invariant subspaces. The opposite coproducts also respect the above diagram, as well as all their Yangian counterparts.

The dimensions of these spaces for generic k, l are $\dim V_{k,l}^I = 1, \dim V_{k,l}^{II} = 4, \dim V_{k,l}^{III} = 6$. For specific values of k, l they can be lower dimensional, e.g. $\dim V_{0,0}^{II} = 2$.

So far we have only used the bosonic part of the algebra to determine the invariant subspaces. The fermionic generators will give maps between the different cases. One can use the (opposite) coproducts of the symmetry generators to move between the different subspaces. In particular, the cases are distinguished by their quantum numbers K^{II}, K^{III} and acting with supersymmetry generators will change these numbers. How this works is schematically depicted in figure 4.1.

These relations between the different cases will play an important role in the derivation of the full S-matrix. We can employ the different arrows in Figure 4.1 (and their Yangian counterparts) to relate the different S-matrices to the Case I S-matrix. In the next section, we will introduce two different sets of bases which allow for a natural interpretation of the S-matrix. These bases will make use of the full Yangian symmetry rather than just \mathfrak{h} . Since we will be able to uniquely determine the form of the S-matrix reduced to Case I states, by applying the aforementioned maps we can use this to compute the S-matrix also in the other cases.

Summarizing, we find that the S-matrix is of block-diagonal form

$$\mathbb{S} = \begin{pmatrix} \boxed{\mathcal{X}} & & & & \\ & \boxed{\mathcal{Y}} & & & \\ & & \boxed{\mathcal{Z}} & & \\ & 0 & & \boxed{\mathcal{Y}} & \\ & & & & \boxed{\mathcal{X}} \end{pmatrix}. \quad (4.17)$$

The outer blocks scatter states from V^I

$$\mathcal{X} : V_N^I \longrightarrow V_N^I \quad (4.18)$$

$$|k, l\rangle^I \mapsto \sum_{m=0}^{k+l} \mathcal{X}_m^{k,l} |m, k+l-m\rangle^I, \quad (4.19)$$

where $k+l=N$ and $\mathcal{X}_m^{k,l}$ will be given by (4.64). The blocks \mathcal{Y} describe the scattering of states from V^{II}

$$\mathcal{Y} : V_N^{II} \longrightarrow V_N^{II} \quad (4.20)$$

$$|k, l\rangle_j^{II} \mapsto \sum_{m=0}^{k+l} \sum_{j=1}^4 \mathcal{Y}_{m;i}^{k,l;j} |m, k+l-m\rangle_j^{II}. \quad (4.21)$$

These S-matrix elements are given in (4.88). Finally, the middle block deals with the third case

$$\mathcal{Z} : V_N^{III} \longrightarrow V_N^{III} \quad (4.22)$$

$$|k, l\rangle_j^{III} \mapsto \sum_{m=0}^{k+l} \sum_{j=1}^6 \mathcal{Z}_{m;i}^{k,l;j} |m, k+l-m\rangle_j^{III}, \quad (4.23)$$

with $\mathcal{Z}_{m;i}^{k,l;j}$ from (4.98).

4.3 Yangian Symmetry and Coproducts

Up to now, we have only used $\mathfrak{su}(2|2)$ symmetry to study the bound state S-matrix. This, however, is not enough to fix the tensor structure of the S-matrix. In particular, it was found that one needs to impose the Yang-Baxter equation by hand to attain this [84]. An alternative to this method was shown to come from Yangian symmetry [85]. We will follow the latter approach and employ Yangian symmetry to fully fix the bound state S-matrix.

4.3.1 (Opposite) coproduct basis

Let us turn back to the invariant subspaces. We define different bases for each case in addition to the standard basis, which is the one commonly used in the literature. These bases are more convenient for the computation of the bound state S-matrix, and they will be called the coproduct basis and the opposite coproduct basis. The basis transformation between the coproduct (opposite coproduct) basis and the standard one will be denoted by Λ (Λ^{op} , respectively).

The (opposite) coproduct basis will be constructed by using Yangian generators to create states out of a chosen vacuum. This is similar to [102] where it was used to study the Bethe Ansatz. We define our vacuum to be

$$|0\rangle \equiv w_1^{\ell_1} v_1^{\ell_2}. \quad (4.24)$$

Note that this state is from V_0^{III} , which is a one dimensional space. The S-matrix maps this space onto itself, and we normalize our S-matrix in such a way that $\mathbb{S}|0\rangle = |0\rangle$. The (opposite) coproduct basis will consist of states created by the (opposite) coproducts of various symmetry generators acting on this vacuum.

Clearly, the S-matrix has a natural interpretation in these bases, and can be formulated in terms of Λ and Λ^{op} , as will be explained below in section 4.3.2. We will now list the explicit formulae for the different bases.

Case I, $K^{\text{II}} = 2N + 2, K^{\text{III}} = N + 2$.

The coproduct basis is given by

$$\Delta(\mathbb{Q}_3^1)\Delta(\mathbb{Q}_2^{\dagger 4}) \prod_{i=q+1}^N \Delta(\mathbb{L}_2^1) \prod_{j=1}^q \Delta(\hat{\mathbb{L}}_2^1)|0\rangle, \quad q = 0, 1, \dots, N, \quad (4.25)$$

and the opposite coproduct basis is given by

$$\Delta^{op}(\mathbb{Q}_3^1)\Delta^{op}(\mathbb{Q}_2^{\dagger 4}) \prod_{i=k+1}^N \Delta^{op}(\mathbb{L}_2^1) \prod_{j=1}^k \Delta^{op}(\hat{\mathbb{L}}_2^1)|0\rangle, \quad k = 0, 1, \dots, N. \quad (4.26)$$

Each of these two bases is indeed composed of $N + 1$ different vectors. By explicitly working out the coproducts one can see that these vectors form a basis for Case I. One could also consider an alternative choice, like for instance

$$\Delta(\mathbb{Q}_3^1)\Delta(\hat{\mathbb{Q}}_3^1) \prod_{i=k+1}^N \Delta(\mathbb{L}_2^1) \prod_{j=1}^k \Delta(\hat{\mathbb{L}}_2^1)|0\rangle, \quad (4.27)$$

but these vectors are readily seen to be proportional to (4.25).

It is also straightforwardly seen why (4.25) actually describes Case I from the point of view of the quantum numbers $K^{\text{II}}, K^{\text{III}}$. The operators $\Delta\mathbb{L}_2^1, \Delta\hat{\mathbb{L}}_2^1$ create a boson of type 2 out of the vacuum and the supersymmetry generators $\Delta\mathbb{Q}_3^1, \Delta\hat{\mathbb{Q}}_3^1$ create a fermion of type 3. Hence we find that $K^{\text{II}} = 2\#\mathbb{L}_2^1 + 2\#\Delta\hat{\mathbb{L}}_2^1 + \#\Delta\mathbb{Q}_3^1 + \#\Delta\hat{\mathbb{Q}}_3^1$ and $K^{\text{III}} = \#\mathbb{L}_2^1 + \#\Delta\hat{\mathbb{L}}_2^1 + \#\Delta\mathbb{Q}_3^1 + \#\Delta\hat{\mathbb{Q}}_3^1$. This indeed coincides with $K^{\text{II}} = 2N + 2, K^{\text{III}} = N + 2$.

Case II, $K^{\text{II}} = 2N + 1, K^{\text{III}} = N + 1$.

The coproduct basis is given by

$$\begin{aligned} \Delta(\mathbb{Q}_3^1) \prod_{i=q+1}^N \Delta(\mathbb{L}_2^1) \prod_{j=1}^q \Delta(\hat{\mathbb{L}}_2^1)|0\rangle, & \quad \Delta(\hat{\mathbb{Q}}_3^1) \prod_{i=q+1}^N \Delta(\mathbb{L}_2^1) \prod_{j=1}^q \Delta(\hat{\mathbb{L}}_2^1)|0\rangle, \\ \Delta(\mathbb{Q}_3^1)\Delta(\hat{\mathbb{Q}}_3^1)\Delta(\mathbb{Q}_4^1) \prod_{i=q+1}^{N-1} \Delta(\mathbb{L}_2^1) \prod_{j=1}^q \Delta(\hat{\mathbb{L}}_2^1)|0\rangle, & \quad \Delta(\mathbb{Q}_3^1)\Delta(\hat{\mathbb{Q}}_3^1)\Delta(\hat{\mathbb{Q}}_4^1) \prod_{i=q+1}^{N-1} \Delta(\mathbb{L}_2^1) \prod_{j=1}^q \Delta(\hat{\mathbb{L}}_2^1)|0\rangle, \end{aligned} \quad (4.28)$$

and similar expressions hold for the opposite coproduct basis. One can again compute $K^{\text{II}}, K^{\text{III}}$ for these states and see explicitly that they describe Case II. Also in this case one could use for example the operator $\Delta(\mathbb{Q}_2^{\dagger 4})$ to create the coproduct basis.

Case III, $K^{\text{II}} = 2N, K^{\text{III}} = N$.

The coproduct basis is

$$\begin{aligned}
& \prod_{i=q+1}^N \Delta(\mathbb{L}_2^1) \prod_{j=1}^q \Delta(\hat{\mathbb{L}}_2^1) |0\rangle, \\
& \Delta(\mathbb{Q}_3^1) \Delta(\mathbb{Q}_4^1) \prod_{i=q+1}^{N-1} \Delta(\mathbb{L}_2^1) \prod_{j=1}^q \Delta(\hat{\mathbb{L}}_2^1) |0\rangle, \\
& \Delta(\mathbb{Q}_3^1) \Delta(\hat{\mathbb{Q}}_4^1) \prod_{i=q+1}^{N-1} \Delta(\mathbb{L}_2^1) \prod_{j=1}^q \Delta(\hat{\mathbb{L}}_2^1) |0\rangle, \\
& \Delta(\hat{\mathbb{Q}}_3^1) \Delta(\mathbb{Q}_4^1) \prod_{i=q+1}^{N-1} \Delta(\mathbb{L}_2^1) \prod_{j=1}^q \Delta(\hat{\mathbb{L}}_2^1) |0\rangle, \\
& \Delta(\hat{\mathbb{Q}}_3^1) \Delta(\hat{\mathbb{Q}}_4^1) \prod_{i=q+1}^{N-1} \Delta(\mathbb{L}_2^1) \prod_{j=1}^q \Delta(\hat{\mathbb{L}}_2^1) |0\rangle, \\
& \Delta(\mathbb{Q}_3^1) \Delta(\mathbb{Q}_4^1) \Delta(\hat{\mathbb{Q}}_3^1) \Delta(\hat{\mathbb{Q}}_4^1) \prod_{i=q+1}^{N-2} \Delta(\mathbb{L}_2^1) \prod_{j=1}^q \Delta(\hat{\mathbb{L}}_2^1) |0\rangle.
\end{aligned}$$

These are readily seen to be $6N$ states and their quantum numbers are of the form $K^{\text{II}} = 2N, K^{\text{III}} = N$.

The Yangian generators also provide maps between the different cases. In particular, one finds that figure 4.1 also holds for Yangian generators. In this basis the arrows between the different cases are obvious. One important thing to notice is the following. Even though, for example, $\Delta\mathbb{Q}_3^2$ maps Case II onto Case I, this does not automatically give a straightforward map between the vector spaces $V_{k,l}^A$. For instance, one has

$$\Delta\mathbb{Q}_3^1 : V_{k,l}^{\text{II}} \longrightarrow V_{k,l-1}^{\text{I}} \oplus V_{k-1,l}^{\text{I}}, \quad (4.29)$$

$$\Delta\hat{\mathbb{Q}}_3^2 : V_{k,l}^{\text{II}} \longrightarrow V_{k+1,l-1}^{\text{I}} \oplus V_{k,l}^{\text{I}} \oplus V_{k-1,l+1}^{\text{I}}. \quad (4.30)$$

This provides an additional complication we will have to deal with in the computation of the S-matrix on Case II states.

4.3.2 S-matrix in coproduct basis

The fact that the coproduct basis is well suited for computing the S-matrix can be seen from (4.6). One sees that the S-matrix directly maps the coproduct basis onto the opposite coproduct

basis. In particular, since we normalize the S-matrix in such a way that $\mathbb{S}|0\rangle = |0\rangle$, we see that the S-matrix, when written as a map between these two bases, is just the identity matrix.

In other words, one can obtain the general formula for the S-matrix in the standard basis (which is ultimately the basis we are interested in) just by applying the appropriate basis transformations. Let us denote the S-matrix written in the standard basis as \mathbb{S} . One then finds:

$$\mathbb{S} = \Lambda^{op} \Lambda^{-1}. \quad (4.31)$$

Since $\Lambda_{12}^{op}(p_1, p_2) = \Lambda_{21}(p_2, p_1)$, this is reminiscent of a Drinfel'd twist [107]. Unfortunately however, the matrix Λ is not of upper triangular form.

Note that the explicit matrices Λ and Λ^{op} just consist of the coproduct vectors written in the standard basis, and that these matrices trivially have the same block structure as the S-matrix with respect to the quantum numbers $K^{\text{II}}, K^{\text{III}}$. The above discussion can be summarized in the following commutative diagram:

$$\begin{array}{ccc} \{\text{coproduct basis}\} & \xrightarrow{1} & \{\text{opposite coproduct basis}\} \\ \Lambda \downarrow & & \Lambda^{op} \downarrow \\ \{\text{standard basis}\} & \xrightarrow{\mathbb{S}} & \{\text{standard basis}\}. \end{array} \quad (4.32)$$

The computationally hard part is finding the explicit inverse of Λ . For any concrete case at hand this can be done by simple linear algebra, but the expressions become rather involved. However, we will be able to carry out this procedure in full generality for the S-matrix of Case I, and use this result to find the S-matrix for all the other states.

4.4 Fundamental S-matrix revisited

To illustrate the above discussion, we will give a full derivation of the fundamental S-matrix \mathbb{S}^F in this formalism. As we saw earlier, the fundamental S-matrix can be completely fixed without usage of Yangian symmetry and we will indeed see this reflected throughout the derivation.

Let us start by explicitly giving the coproduct basis and the standard basis. The vacuum is given by

$$|0\rangle \in V_0^{\text{III}}, \quad |0\rangle = w_1 v_1. \quad (4.33)$$

We choose the normalization $\mathbb{S}^F|0\rangle = |0\rangle$.

It is readily seen that there is only one Case I state, namely

$$\theta_3 \vartheta_3. \quad (4.34)$$

The corresponding state from Case Ib is found by replacing $3 \leftrightarrow 4$ and does not give anything new. The (opposite) coproduct basis for this case is given by

$$\Delta Q_3^1 \Delta Q_2^{\dagger 4} |0\rangle, \quad \Delta^{op} Q_3^1 \Delta^{op} Q_2^{\dagger 4} |0\rangle. \quad (4.35)$$

One can explicitly work out the above to find

$$\Delta Q_3^1 \Delta Q_2^{\dagger 4} |0\rangle = (a_1 c_2 - a_2 c_1) \theta_3 \vartheta_3. \quad (4.36)$$

Hence, the piece of Λ that describes the basis transformation of Case III states is given by

$$\Lambda_{\mathcal{X}} = a_1 c_2 - a_2 c_1. \quad (4.37)$$

$$\Lambda_{\mathcal{X}}^{op} = a_3 c_4 - a_4 c_3. \quad (4.38)$$

To avoid cluttered notation, we introduce the following quantities

$$\begin{aligned} \mathcal{Q}_{ij} &= a_i c_j - a_j c_i, \\ \overline{\mathcal{Q}}_{ij} &= b_i d_j - d_j b_i, \\ \mathcal{J}_{ij} &= a_i d_j - b_j c_i. \end{aligned} \quad (4.39)$$

These coefficients satisfy the following identity

$$\mathcal{Q}_{ij} \overline{\mathcal{Q}}_{ij} = 1 - \mathcal{J}_{ij} \mathcal{J}_{ji}. \quad (4.40)$$

We also work with coefficients that carry the explicit braiding factors (3.62, 3.63) to keep the notation light. The S-matrix restricted to Case I follows readily from this,

$$\mathbb{S}^F \cdot \theta_3 \theta_3 = \mathcal{X} \theta_3 \theta_3, \quad (4.41)$$

with

$$\mathcal{X} = \frac{a_3 c_4 - a_4 c_3}{a_1 c_2 - a_2 c_1} = \frac{\mathcal{Q}_{34}}{\mathcal{Q}_{12}}. \quad (4.42)$$

Next, we consider Case II. There are four different invariant subspaces, but upon changing $3 \leftrightarrow 4, 1 \leftrightarrow 2$ we find that they are all isomorphic. We will therefore restrict to V_0^{II} , which has standard basis

$$\{\theta_3 v_1, w_1 \vartheta_3\}. \quad (4.43)$$

The coproduct basis is easily seen to be given by

$$\{\Delta Q_3^1 |0\rangle, \Delta Q_2^{\dagger 4} |0\rangle\}. \quad (4.44)$$

Again, we would like to point out that

$$\{\Delta Q_3^1|0\rangle, \Delta \hat{Q}_3^1|0\rangle\} \quad (4.45)$$

is also a valid basis, and both bases will necessarily result in the same S-matrix. The coproduct-to-standard basis transformation can be worked out to find, in the above notation,

$$\Lambda_{\mathcal{Y}} = \begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \end{pmatrix}. \quad (4.46)$$

The S-matrix for Case II is given by

$$\begin{aligned} \mathbb{S}^F \cdot \theta_3 v_1 &= \mathcal{Y}_1^1 \theta_3 v_1 + \mathcal{Y}_1^2 w_1 \vartheta_3 \\ \mathbb{S}^F \cdot w_1 \vartheta_3 &= \mathcal{Y}_2^1 \theta_3 v_1 + \mathcal{Y}_2^2 w_1 \vartheta_3, \end{aligned} \quad (4.47)$$

with

$$\mathcal{Y} = \begin{pmatrix} \mathcal{Y}_1^1 & \mathcal{Y}_1^2 \\ \mathcal{Y}_2^1 & \mathcal{Y}_2^2 \end{pmatrix} = \begin{pmatrix} \frac{a_2 c_3 - a_3 c_2}{a_2 c_1 - a_1 c_2} & \frac{a_3 c_1 - a_1 c_3}{a_2 c_1 - a_1 c_2} \\ \frac{a_2 c_4 - c_2 a_4}{a_2 c_1 - a_1 c_2} & \frac{a_4 c_1 - a_1 c_4}{a_2 c_1 - a_1 c_2} \end{pmatrix} = \begin{pmatrix} \frac{\mathcal{Q}_{23}}{\mathcal{Q}_{21}} & \frac{\mathcal{Q}_{31}}{\mathcal{Q}_{21}} \\ \frac{\mathcal{Q}_{24}}{\mathcal{Q}_{21}} & \frac{\mathcal{Q}_{41}}{\mathcal{Q}_{21}} \end{pmatrix}. \quad (4.48)$$

Finally, for Case III one finds three different subspaces $V_0^{\text{III}}, V_1^{\text{III}}, V_2^{\text{III}}$. The subspace V_0^{III} contains only the vacuum (4.33), and V_2^{III} is isomorphic to V_0^{III} . The only non-trivial piece is V_1^{III} which is spanned by the standard basis

$$\{w_1 v_2, w_2 v_1, \theta_3 \vartheta_4, \theta_4 \vartheta_3\}. \quad (4.49)$$

The coproduct basis is given by

$$\{\Delta \mathbb{L}_2^1|0\rangle, \Delta Q_3^1 \Delta Q_4^1|0\rangle, \Delta Q_3^1 \Delta Q_2^{\dagger 3}|0\rangle, \Delta Q_2^{\dagger 4} \Delta Q_2^{\dagger 3}|0\rangle\} \quad (4.50)$$

and from this one finds

$$\Lambda_{\mathcal{X}} = \begin{pmatrix} 1 & -a_2 b_2 & a_2 d_2 & -c_2 d_2 \\ 1 & -a_1 b_1 & a_1 d_1 & -c_1 d_1 \\ 0 & -a_1 a_2 & a_1 c_2 & -c_1 c_2 \\ 0 & -a_1 a_2 & a_2 c_1 & -c_1 c_2 \end{pmatrix}. \quad (4.51)$$

The resulting S-matrix for Case III is

$$\mathcal{S} = \Lambda_{\mathcal{X}}^{op} (\Lambda_{\mathcal{X}})^{-1} = \begin{pmatrix} \frac{\mathcal{Q}_{14} \mathcal{I}_{14}}{\mathcal{Q}_{12} \mathcal{I}_{12}} & \frac{\mathcal{Q}_{24} \mathcal{I}_{24}}{\mathcal{Q}_{21} \mathcal{I}_{21}} & \frac{\mathcal{Q}_{24} \overline{\mathcal{Q}}_{14}}{\mathcal{Q}_{12} \mathcal{I}_{21}} & \frac{\mathcal{Q}_{14} \overline{\mathcal{Q}}_{42}}{\mathcal{Q}_{21} \mathcal{I}_{12}} \\ \frac{\mathcal{Q}_{13} \mathcal{I}_{13}}{\mathcal{Q}_{12} \mathcal{I}_{12}} & \frac{\mathcal{Q}_{23} \mathcal{I}_{23}}{\mathcal{Q}_{21} \mathcal{I}_{21}} & \frac{\mathcal{Q}_{23} \mathcal{I}_{13}}{\mathcal{Q}_{12} \mathcal{I}_{21}} & \frac{\mathcal{Q}_{13} \mathcal{I}_{32}}{\mathcal{Q}_{21} \mathcal{I}_{12}} \\ \frac{\mathcal{Q}_{13} \mathcal{Q}_{14}}{\mathcal{Q}_{12} \mathcal{I}_{12}} & \frac{\mathcal{Q}_{23} \mathcal{Q}_{24}}{\mathcal{Q}_{21} \mathcal{I}_{21}} & -\frac{\mathcal{Q}_{23} \mathcal{I}_{41}}{\mathcal{Q}_{12} \mathcal{I}_{21}} & \frac{\mathcal{Q}_{13} \mathcal{I}_{42}}{\mathcal{Q}_{21} \mathcal{I}_{12}} \\ -\frac{\mathcal{Q}_{13} \mathcal{Q}_{14}}{\mathcal{Q}_{12} \mathcal{I}_{12}} & -\frac{\mathcal{Q}_{23} \mathcal{Q}_{24}}{\mathcal{Q}_{21} \mathcal{I}_{21}} & \frac{\mathcal{Q}_{24} \mathcal{I}_{31}}{\mathcal{Q}_{12} \mathcal{I}_{21}} & -\frac{\mathcal{Q}_{14} \mathcal{I}_{32}}{\mathcal{Q}_{21} \mathcal{I}_{12}} \end{pmatrix}. \quad (4.52)$$

This is a nice expression which only depend on the representation parameters a, b, c, d which automatically incorporate the braiding factors. If one plugs in the explicit parameterizations in terms of x^\pm (3.62, 3.63) one recovers the fundamental S-matrix (4.1). It is also readily seen that using Yangian generators to construct the (opposite) coproduct basis leads to the same S-matrix.

Writing Λ as a 16×16 -matrix one finds a decomposition of \mathbb{S}^F which reminds of a Drinfel'd twist:

$$\mathbb{S}^F = \Lambda^{op} \Lambda^{-1}. \quad (4.53)$$

The inverse S-matrix is easily found by interchanging the opposite and normal coproduct basis. This just amounts to changing

$$(a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2) \leftrightarrow (a_3, b_3, c_3, d_3, a_4, b_4, c_4, d_4), \quad (4.54)$$

in the above formulae. In the previous discussion there was no need to use the Yangian generators, however they will prove crucial for general bound states.

4.5 Complete Solution of Case I

We will now move on to generic bound state representations. As mentioned before we will first derive the S-matrix for states from Case I. To this end, we employ Yangian symmetry. We will work in the evaluation representation. This S-matrix proves to be the building block out of which the S-matrices for both Case II and Case III can be constructed¹.

Our starting point is the coproduct basis (4.25). It is convenient to reorder the products in the following way:

$$\left\{ \prod_{i=q+1}^N \Delta(\mathbb{L}_2^1) \prod_{j=1}^q \Delta(\hat{\mathbb{L}}_2^1) \right\} \Delta(\mathbb{Q}_3^1) \Delta(\mathbb{Q}_2^{\dagger 4}) |0\rangle. \quad (4.55)$$

The action of the susy generators on the vacuum is of the form

$$\Delta(\mathbb{Q}_3^1) \Delta(\mathbb{Q}_2^{\dagger 4}) |0\rangle = (a_2 c_1 - a_1 c_2) \ell_1 \ell_2 |0, 0\rangle^I, \quad (4.56)$$

with a similar expression for the opposite version. In complete analogy with the fundamental S-matrix, this defines the coordinate transformation Λ of Case I states to the standard basis,

¹Our procedure will somehow be reminiscent of employing highest weight states of Yangians.

from which one can straightforwardly read off the action of the S-matrix on $|0, 0\rangle^I$:

$$\begin{aligned} \mathbb{S}|0, 0\rangle^I &= \frac{\mathbb{S}\Delta(\mathbb{Q}_3^1)\Delta(\mathbb{Q}_2^{\dagger 4})|0\rangle}{(a_2c_1 - a_1c_2)\ell_1\ell_2} \\ &= \frac{\Delta^{op}(\mathbb{Q}_3^1)\Delta^{op}(\mathbb{Q}_2^{\dagger 4})\mathbb{S}|0\rangle}{(a_2c_1 - a_1c_2)\ell_1\ell_2} \\ &= \frac{a_4c_3 - a_3c_4}{a_2c_1 - a_1c_2}|0, 0\rangle^I. \end{aligned} \quad (4.57)$$

In other words, the S-matrix multiplies $|0, 0\rangle^I$ by a scalar. We will denote this scalar by \mathcal{D} . In terms of x^\pm , it is given by

$$\mathcal{D} \equiv \frac{a_4c_3 - a_3c_4}{a_2c_1 - a_1c_2} = \frac{x_2^+ - x_1^-}{x_2^- - x_1^+} \sqrt{\frac{x_1^+}{x_1^-}} \sqrt{\frac{x_2^-}{x_2^+}}. \quad (4.58)$$

This factor coincides with the one found for the fundamental S-matrix on the corresponding state.

One can now use the generators $\mathbb{L}_2^1, \hat{\mathbb{L}}_2^1$ to construct a generic Case I state $|k, l\rangle^I$ from $|0, 0\rangle^I$, for arbitrary k, l . This can be seen by considering the following identities:

$$\begin{aligned} (\mathbb{L}_2^1 \otimes \mathbb{1})(\delta u + \Delta\mathbb{L}_1^1)|k, l\rangle^I &= \left\{ \Delta\hat{\mathbb{L}}_2^1 - u_2\Delta\mathbb{L}_2^1 + \Delta\mathbb{L}_2^1 \circ (\mathbb{L}_1^1 \otimes \mathbb{1}) \right\} |k, l\rangle^I, \\ (\mathbb{1} \otimes \mathbb{L}_2^1)(\delta u + \Delta\mathbb{L}_1^1)|k, l\rangle^I &= - \left\{ \Delta\hat{\mathbb{L}}_2^1 - u_1\Delta\mathbb{L}_2^1 - \Delta\mathbb{L}_2^1 \circ (\mathbb{1} \otimes \mathbb{L}_1^1) \right\} |k, l\rangle^I, \end{aligned} \quad (4.59)$$

where

$$\delta u = u_1 - u_2. \quad (4.60)$$

Since $\Delta(\mathbb{L}_1^1)|k, l\rangle^I = \frac{\ell_1 + \ell_2 - 2(k+l+1)}{2}|k, l\rangle^I$, it is obvious that the left hand side of (4.59) is proportional to $|k+1, l\rangle$ (first line) and $|k, l+1\rangle$ (respectively, second line). By applying the right hand side operators in (4.59) inductively to $|0, 0\rangle$, one finds

$$\begin{aligned} \left\{ \prod_{m=1}^k (\ell_1 - m) \prod_{n=1}^l (\ell_2 - n) \prod_{q=1}^{k+l} \left(\delta u + \frac{\ell_1 + \ell_2}{2} - q \right) \right\} |k, l\rangle &= \\ [(\mathbb{L}_2^1 \otimes \mathbb{1})(\delta u + \Delta\mathbb{L}_1^1)]^k [(\mathbb{1} \otimes \mathbb{L}_2^1)(\delta u + \Delta\mathbb{L}_1^1)]^l |0, 0\rangle. \end{aligned} \quad (4.61)$$

Then, by (4.59),

$$|k, l\rangle^I = \frac{\prod_{i=1}^k \left[\Delta\hat{\mathbb{L}}_2^1 - \frac{\ell_1 + 2u_2 - 2i + 1}{2} \Delta\mathbb{L}_2^1 \right] \prod_{j=1}^l \left[\frac{1 + 2j + 2u_1 - \ell_2}{2} \Delta\mathbb{L}_2^1 - \Delta\hat{\mathbb{L}}_2^1 \right]}{\prod_{m=1}^k (\ell_1 - m) \prod_{n=1}^l (\ell_2 - n) \prod_{q=1}^{k+l} \left(\delta u + \frac{\ell_1 + \ell_2}{2} - q \right)} |0, 0\rangle^I.$$

This exactly tells us how to write a state in the standard basis in terms of the coproduct basis. In other words, this explicitly indicates how to construct Λ^{-1} . We would also like to point out

that the presence of the full Yangian symmetry is crucial here. It is not possible to construct the operators that link the vectors $|k, l\rangle^I$ to $|0, 0\rangle^I$ without the Yangian coproducts.

It is now straightforward to obtain the action of the S-matrix on Case I states from the above. The symmetry properties of the S-matrix, together with (4.57), now imply

$$\mathbb{S}|k, l\rangle^I = \mathcal{D} \times \frac{\prod_{i=1}^k \left[\Delta^{op} \hat{\mathbb{L}}_2^1 - \frac{2u_2 - \ell_1 + 2i - 1}{2} \Delta^{op} \mathbb{L}_2^1 \right] \prod_{j=1}^l \left[\frac{2u_1 + \ell_2 - 1 - 2j}{2} \Delta^{op} \mathbb{L}_2^1 - \Delta^{op} \hat{\mathbb{L}}_2^1 \right]}{\prod_{m=1}^k (\ell_1 - m) \prod_{n=1}^l (\ell_2 - n) \prod_{q=1}^{k+l} \left(\delta u + \frac{\ell_1 + \ell_2}{2} - q \right)} |0, 0\rangle^I. \quad (4.62)$$

By explicitly computing the right hand side, one finds that $\mathbb{S}|k, l\rangle^I$ is of the form

$$\mathbb{S}|k, l\rangle^I = \sum_{n=0}^{k+l} \mathcal{X}_n^{k,l} |n, k+l-n\rangle^I, \quad (4.63)$$

with

$$\begin{aligned} \mathcal{X}_n^{k,l} = \mathcal{D} & \frac{\prod_{i=1}^n (\ell_1 - i) \prod_{i=1}^{k+l-n} (\ell_2 - i)}{\prod_{p=1}^k (\ell_1 - p) \prod_{p=1}^l (\ell_2 - p) \prod_{p=1}^{k+l} \left(\delta u + \frac{\ell_1 + \ell_2}{2} - p \right)} \times \\ & \times \sum_{m=0}^k \left\{ \binom{k}{k-m} \binom{l}{n-m} \prod_{p=1}^m \mathfrak{c}_p^+ \prod_{p=1-m}^{l-n} \mathfrak{c}_p^- \prod_{p=1}^{k-m} \mathfrak{d}_{\frac{k-p+2}{2}} \prod_{p=1}^{n-m} \tilde{\mathfrak{d}}_{\frac{k+l-m-p+2}{2}} \right\}. \end{aligned} \quad (4.64)$$

The coefficients are given by

$$\begin{aligned} \mathfrak{c}_m^\pm &= \delta u \pm \frac{\ell_1 - \ell_2}{2} - m + 1, \\ \mathfrak{d}_i &= \ell_1 + 1 - 2i, \\ \tilde{\mathfrak{d}}_i &= \ell_2 + 1 - 2i. \end{aligned}$$

It is worthwhile noticing that in the special case $l = 0$ (and similarly for $k = 0$) this expression reduces considerably. For later use, we can write it in the following way:

$$\mathcal{X}_{k-n}^{k,0} = \mathcal{D} D \binom{k}{n} \frac{\prod_{p=1}^n (\ell_2 - p) \prod_{p=1}^{k-n} \left(\delta u + \frac{\ell_1 - \ell_2}{2} - p + 1 \right)}{\prod_{p=1}^k \left(\delta u + \frac{\ell_1 + \ell_2}{2} - p \right)}. \quad (4.65)$$

In all of the above expressions it is understood that products are set to 1 whenever they run over negative integers, i.e. $\prod_a^b = 1$ if $b < a$, and the binomial $\binom{x}{y}$ is taken to be zero if $y > x$ and if $y < 0$.

We can see how the formula we have found bears a rational dependence on the difference of the spectral parameters, as typical of Yangian universal R-matrices in evaluation representations cf. e.g. [109, 110]. The following function, meromorphic in all the parameters, coincides with

(4.64) in the appropriate domain of integer values:

$$\begin{aligned} \mathcal{X}_n^{k,l} &= \frac{(-1)^{k+n} \pi \mathcal{D} \sin[(k - \ell_1)\pi] \Gamma(l+1)}{\sin[\ell_1 \pi] \sin[(k + l - \ell_2 - n)\pi] \Gamma(l - \ell_2 + 1) \Gamma(n+1)} \times \\ &\quad \frac{\Gamma(n+1 - \ell_1) \Gamma\left(l + \frac{\ell_1 - \ell_2}{2} - n - \delta u\right) \Gamma\left(1 - \frac{\ell_1 + \ell_2}{2} - \delta u\right)}{\Gamma\left(k + l - \frac{\ell_1 + \ell_2}{2} - \delta u + 1\right) \Gamma\left(\frac{\ell_1 - \ell_2}{2} - \delta u\right)} \times \\ &\quad {}_4\tilde{F}_3 \left[-k, -n, \frac{2\delta u + 2 - \ell_1 + \ell_2}{2}, \frac{\ell_2 - \ell_1 - 2\delta u}{2}; 1 - \ell_1, \ell_2 - k - l, l - n + 1; 1 \right], \end{aligned} \quad (4.66)$$

where one has defined the regularized hypergeometric function ${}_4\tilde{F}_3(a_1, a_2, a_3, a_4; b_1, b_2, b_3; \tau) = {}_4F_3(a_1, a_2, a_3, a_4; b_1, b_2, b_3; \tau) / [\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)]$.

Moreover, we can easily see that we are in a special situation, since the parameters entering the hypergeometric function ${}_4F_3(a_1, a_2, a_3, a_4; b_1, b_2, b_3; 1)$ satisfy $\sum_{i=1}^4 a_i - \sum_{j=1}^3 b_j = -1$. When this happens, the hypergeometric function reduces to a $6j$ -symbol, according to the following formula (see for example [136]):

$$\begin{aligned} {}_4F_3(a_1, a_2, a_3, a_4; b_1, b_2, b_3; 1) &= \\ &\quad \frac{(-1)^{b_1+1} \Gamma(b_2) \Gamma(b_3) \sqrt{\Gamma(1-a_1) \Gamma(1-a_2) \Gamma(1-a_3)}}{\Gamma(1-b_1) \sqrt{\Gamma(b_2-a_1) \Gamma(b_2-a_2)}} \times \\ &\quad \times \frac{\sqrt{\Gamma(1-a_4) \Gamma(a_1-b_1+1) \Gamma(a_2-b_1+1) \Gamma(a_3-b_1+1) \Gamma(a_4-b_1+1)}}{\sqrt{\Gamma(b_2-a_3) \Gamma(b_2-a_4) \Gamma(b_3-a_1) \Gamma(b_3-a_2) \Gamma(b_3-a_3) \Gamma(b_3-a_4)}} \times \\ &\quad \times \left\{ \begin{array}{ccc} \frac{1}{2}(-a_1 - a_4 + b_3 - 1) & \frac{1}{2}(-a_1 - a_3 + b_2 - 1) & \frac{1}{2}(a_1 + a_2 - b_1 - 1) \\ \frac{1}{2}(-a_2 - a_3 + b_3 - 1) & \frac{1}{2}(-a_2 - a_4 + b_2 - 1) & \frac{1}{2}(a_3 + a_4 - b_1 - 1) \end{array} \right\}. \end{aligned} \quad (4.67)$$

By identifying the parameters we see that the relevant $6j$ -symbol

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} \quad (4.68)$$

has coefficients

$$\begin{aligned} j_1 &= \frac{1}{2} \left(k + l - n + \frac{\ell_1 - \ell_2}{2} + \delta u \right), & j_2 &= \frac{1}{2} \left(\frac{\ell_1 + \ell_2}{2} - 2 - l - \delta u \right), \\ j_3 &= \frac{1}{2} (\ell_1 - 2 - k - n), & j_4 &= \frac{1}{2} \left(\frac{\ell_1 - \ell_2}{2} - 1 + l - \delta u \right), \\ j_5 &= \frac{1}{2} \left(\frac{\ell_1 + \ell_2}{2} - 1 - k - l + n + \delta u \right), & j_6 &= \frac{1}{2} (\ell_2 - 1). \end{aligned} \quad (4.69)$$

For generic values of δu , the $6j$ -symbol is understood in the same sense as in the comment above formula (4.66). However, one can prove that, for values of δu corresponding to the physical poles, the entries of the $6j$ -symbol are indeed half-integer, as one may expect from the fusion rules of $\mathfrak{su}(2)$ representations. One expects this because the action of the bosonic $\mathfrak{su}(2)$ generators \mathbb{L}_b^a on Case I states forms a $\mathfrak{su}(2)$ algebra.

In the special case $l = 0$ (a similar argument would hold for $k = 0$), we can go back to expression (4.64), and see that it can be casted in the following form:

$$\mathcal{X}_{k-n}^{k,0} = \frac{\mathcal{D}\Gamma(k+1)\Gamma(1+n-\ell_2)\Gamma\left(1-\frac{\ell_1+\ell_2}{2}-\delta u\right)\Gamma\left(k+\frac{\ell_2}{2}-\frac{\ell_1}{2}-n-\delta u\right)}{\Gamma(1-\ell_2)\Gamma(k-n+1)\Gamma(n+1)\Gamma\left(k-\frac{\ell_1+\ell_2}{2}-\delta u+1\right)\Gamma\left(\frac{\ell_2-\ell_1}{2}-\delta u\right)}. \quad (4.71)$$

4.6 The S-matrix for Case II

As explained in the previous sections, $\Delta\mathbb{Q}_3^1, \Delta\mathbb{Q}_2^{\dagger 4}$ and their Yangian counterparts map Case II states onto Case I states. We introduce the Case II S-matrix in the following way

$$\mathbb{S}|k, l\rangle_i^{\text{II}} = \sum_{j=1}^4 \sum_{m=0}^{k+l} \mathcal{Y}_{m;i}^{k,l;j} |m, N-m\rangle_j^{\text{II}}, \quad (4.72)$$

where again $N = k + l$. This means that the coefficients $\mathcal{Y}_{m;i}^{k,l;j}$ actually correspond to the S-matrix restricted to the following spaces

$$\mathcal{Y}_{n;i}^{k,l;j} : V_{k,l}^{\text{II}} \longrightarrow V_{n,N-n}^{\text{II}}. \quad (4.73)$$

Generically, both spaces are 4 dimensional, and $\mathcal{Y}_{m;i}^{k,l;j}$ correspond to the coefficients of a 4×4 matrix. One might wonder what happens for special values of k, l, n, N since V_0^{II} is lower dimensional. It turns out that the 4×4 matrix actually contains these non-generic cases. This will be explained later on in Section 4.8 and we will continue with deriving the generic 4×4 matrix.

By considering the action of $\Delta\mathbb{Q}_3^1$, we can relate the Case II S-matrix to (4.64). It is easily checked that

$$\Delta\mathbb{Q}_3^1 |k, l\rangle_j^{\text{II}} = Q_j(k, l) |k, l\rangle_j^{\text{I}}, \quad (4.74)$$

with

$$\begin{aligned} Q_1(k, l) &= a_2(l - \ell_2), & Q_2(k, l) &= a_1(\ell_1 - k), \\ Q_3(k, l) &= b_2, & Q_4(k, l) &= -b_1. \end{aligned} \quad (4.75)$$

Similar expressions are of course obtained for $\Delta^{op}\mathbb{Q}_3^1, \Delta^{op}\mathbb{Q}_2^{\dagger 4}, \Delta\mathbb{Q}_2^{\dagger 4}$. We can now apply our general strategy in the following fashion:

$$\begin{aligned} {}^{\text{I}}\langle n, N-n | \Delta^{op}\mathbb{Q}_3^1 \mathbb{S} |k, l\rangle_i^{\text{II}} &= \sum_{j=1}^4 \sum_{m=0}^{k+l} \mathcal{Y}_{m;i}^{k,l;j} {}^{\text{I}}\langle n, N-n | \Delta^{op}\mathbb{Q}_3^1 |m, N-m\rangle_j^{\text{II}} \\ &= \sum_{j,m} \mathcal{Y}_{m;i}^{k,l;j} Q_j^{op}(m, N-m) {}^{\text{I}}\langle n, N-n | m, N-m \rangle_j^{\text{I}} \\ &= \sum_{j=1}^4 \mathcal{Y}_{n;i}^{k,l;j} Q_j^{op}(n, N-n). \end{aligned} \quad (4.76)$$

On the other hand, we can use the symmetry properties of the S-matrix to obtain

$$\begin{aligned}
{}^I\langle n, N-n | \Delta^{op} \mathbb{Q}_3^1 \mathbb{S} | k, l \rangle_i^\Pi &= {}^I\langle n, N-n | \mathbb{S} \Delta \mathbb{Q}_3^1 | k, l \rangle_i^\Pi \\
&= Q_i(k, l) {}^I\langle n, N-n | \mathbb{S} | k, l \rangle^I \\
&= Q_i(k, l) \sum_{m=0}^N \mathcal{X}_m^{k,l} {}^I\langle n, N-n | m, N-m \rangle^I \\
&= Q_i(k, l) \mathcal{X}_n^{k,l}.
\end{aligned} \tag{4.77}$$

Clearly, this gives us four linear equations relating the S-matrix from Case II to the S-matrix of Case I. A similar computation can be worked out using $\Delta^{op} \mathbb{Q}_2^{\dagger 4}$, giving four additional equations. We can cast the above formulae in a convenient matrix form:

$$\begin{pmatrix} a_4(N-n-\ell_2) & a_3(\ell_1-n) & b_4 & -b_3 \\ c_4(N-n-\ell_2) & c_3(\ell_1-n) & d_4 & -d_3 \end{pmatrix} \mathcal{Y}_n^{k,l} = \mathcal{X}_n^{k,l} \begin{pmatrix} a_2(l-\ell_2) & a_1(\ell_1-k) & b_2 & -b_1 \\ c_2(l-\ell_2) & c_1(\ell_1-k) & d_2 & -d_1 \end{pmatrix}, \tag{4.78}$$

with

$$\mathcal{Y}_n^{k,l} \equiv \begin{pmatrix} \mathcal{Y}_{n;1}^{k,l;1} & \mathcal{Y}_{n;2}^{k,l;1} & \mathcal{Y}_{n;3}^{k,l;1} & \mathcal{Y}_{n;4}^{k,l;1} \\ \mathcal{Y}_{n;1}^{k,l;2} & \mathcal{Y}_{n;2}^{k,l;2} & \mathcal{Y}_{n;3}^{k,l;2} & \mathcal{Y}_{n;4}^{k,l;2} \\ \mathcal{Y}_{n;1}^{k,l;3} & \mathcal{Y}_{n;2}^{k,l;3} & \mathcal{Y}_{n;3}^{k,l;3} & \mathcal{Y}_{n;4}^{k,l;3} \\ \mathcal{Y}_{n;1}^{k,l;4} & \mathcal{Y}_{n;2}^{k,l;4} & \mathcal{Y}_{n;3}^{k,l;4} & \mathcal{Y}_{n;4}^{k,l;4} \end{pmatrix}. \tag{4.79}$$

Written in this way, the relation to (4.6) becomes apparent. However, because the matrix $\mathcal{Y}_n^{k,l}$ has 16 unknown coefficients it is clear that in order to fully determine $\mathcal{Y}_n^{k,l}$ (and therefore the full Case II S-matrix) one needs more equations in addition to (4.78).

These equations can be obtained via the Yangian generators. Consider the following operators:

$$\Delta \tilde{\mathbb{Q}} = \Delta \hat{\mathbb{Q}}_3^1 + \frac{\Delta \hat{\mathbb{L}}_2^1 \Delta \mathbb{Q}_3^2}{\frac{\ell_1+\ell_2}{2} - (N+1+\delta u)} - \frac{\frac{\ell_1-\ell_2}{2} + N - 2n + u_1 + u_2}{\ell_1 + \ell_2 - (N+1+\delta u)} \Delta \mathbb{L}_2^1 \Delta \mathbb{Q}_3^2, \tag{4.80}$$

$$\Delta \tilde{\mathbb{G}} = \Delta \hat{\mathbb{Q}}_2^{\dagger 4} + \frac{\Delta \hat{\mathbb{L}}_2^1 \Delta \mathbb{Q}_1^{\dagger 4}}{\frac{\ell_1+\ell_2}{2} - (N+1+\delta u)} + \frac{\frac{\ell_1-\ell_2}{2} + N - 2n + u_1 + u_2}{\ell_1 + \ell_2 - 2(N+1+\delta u)} \Delta \mathbb{L}_2^1 \Delta \mathbb{Q}_1^{\dagger 4}.$$

These operators are chosen in such a way that *only* states of the form $|n, N-n\rangle_i^\Pi$ are mapped to $|n, N-n\rangle_i^I$. When we follow the same derivation as before, we see that this fact is important in (4.76) in order to be able to factorize the matrix $\mathcal{Y}_n^{k,l}$ in front of the final expression, and be therefore able to solve for it. In fact, $\Delta \tilde{\mathbb{Q}}$ generically maps

$$\Delta \tilde{\mathbb{Q}} : V_{k,l}^\Pi \longrightarrow V_{k+1,l-1}^I \oplus V_{k,l}^I \oplus V_{k-1,l+1}^I, \tag{4.81}$$

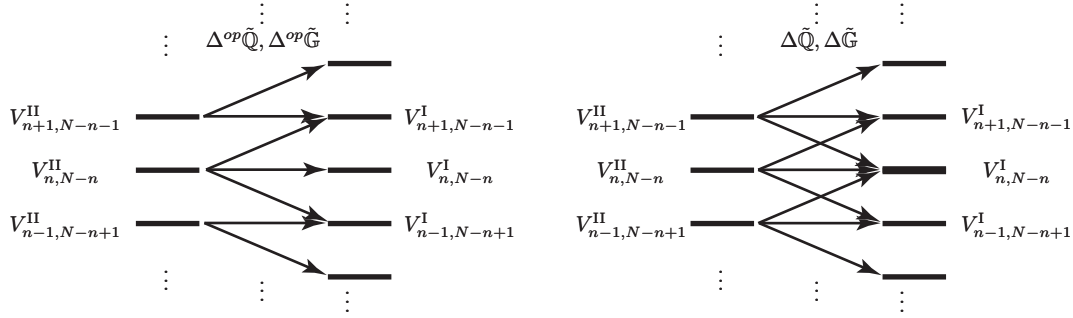


Figure 4.2: Action of $\Delta^{op}\tilde{Q}$, $\Delta^{op}\tilde{G}$ and $\Delta\tilde{Q}$, $\Delta\tilde{G}$. They map Case II states (on the left) to Case I states (on the right).

or, more precisely, we can write

$$\Delta\tilde{Q}|k, l\rangle_i^{\text{II}} = \tilde{Q}_i(k, l)|k, l\rangle^{\text{I}} + \tilde{Q}_i^+(k+1, l-1)|k+1, l-1\rangle^{\text{I}} + \tilde{Q}_i^-(k-1, l+1)|k-1, l+1\rangle^{\text{I}}. \quad (4.82)$$

This means that, if one follows (4.76), one obtains

$${}^{\text{I}}\langle n, N-n | \Delta^{op}\tilde{Q}S | k, l \rangle_i^{\text{II}} = \sum_{j=1}^4 \mathcal{Y}_{n,i}^{k,l;j} \tilde{Q}_j^{op}(n, N-n) + \mathcal{Y}_{n+1,i}^{k,l;j} \tilde{Q}_j^{op,+}(n, N-n) + \mathcal{Y}_{n-1,i}^{k,l;j} \tilde{Q}_j^{op,-}(n, N-n). \quad (4.83)$$

However, the specific choice we made for $\Delta\tilde{Q}$ means that $\tilde{Q}_j^{op,+}(n, N-n) = \tilde{Q}_j^{op,-}(n, N-n) = 0$. In other words, we can again put in evidence the matrix factor $\mathcal{Y}_n^{k,l}$ on the left hand side of the final equation. Since this is specifically tuned to work for the opposite coproducts, the right hand side of the equation will not have this property, and \tilde{Q}^{\pm} will contribute there. This is exemplified in figure 4.2.

For compactness, let us define $M \equiv N - 2n$. By combining all the equations one is lead to the following matrix equation:

$$\begin{pmatrix} a_4 & a_3 & 0 & 0 \\ c_4 & c_3 & 0 & 0 \\ 0 & 0 & a_4 & a_3 \\ 0 & 0 & c_4 & c_3 \end{pmatrix} A \mathcal{Y}_n^{k,l} = \begin{pmatrix} a_2 & a_1 & 0 & 0 \\ c_2 & c_1 & 0 & 0 \\ 0 & 0 & a_2 & a_1 \\ 0 & 0 & c_2 & c_1 \end{pmatrix} \left[B^+ \mathcal{X}_n^{k+1,l-1} + B^- \mathcal{X}_n^{k-1,l+1} + B \mathcal{X}_n^{k,l} \right] \quad (4.84)$$

where the matrix on the left hand side is given by

$$A = \begin{pmatrix} N-n-\ell_2 & 0 & \frac{\mathcal{I}_{34}}{\mathcal{Q}_{34}} & \frac{1}{\mathcal{Q}_{43}} \\ 0 & \ell_1-n & \frac{1}{\mathcal{Q}_{43}} & \frac{\mathcal{I}_{43}}{\mathcal{Q}_{34}} \\ (N-n-\ell_2)(M-\delta u) & (n-\ell_1)\ell_2\mathcal{I}_{34} & \frac{(\delta u-M+\ell_2)\mathcal{I}_{34}}{\mathcal{Q}_{43}} & \frac{\delta u+M+\ell_1-\ell_2}{\mathcal{Q}_{43}} \frac{\mathcal{Q}_{34}\overline{\mathcal{Q}}_{34}}{\mathcal{Q}_{43}} \\ (N-n-\ell_2)(\ell_1\mathcal{I}_{43}) & (\ell_1-n)(\delta u+M) & \frac{M-\delta u-\ell_2+\ell_1}{\mathcal{Q}_{43}} \frac{\mathcal{Q}_{34}\overline{\mathcal{Q}}_{34}}{\mathcal{Q}_{43}} & \frac{(\delta u+M+\ell_1)\mathcal{I}_{43}}{\mathcal{Q}_{34}} \end{pmatrix} \quad (4.85)$$

and the matrices on the right hand side by

$$\begin{aligned}
 B^+ &= \frac{2(\ell_1-k-1)\mathfrak{c}_{l-n}^-}{\tilde{\mathfrak{c}}_{-N}^-} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ l & 0 & \frac{\mathcal{I}_{12}}{\mathcal{Q}_{12}} & 0 \\ 0 & 0 & \frac{1}{\mathcal{Q}_{21}} & 0 \end{pmatrix}, & B^- &= \frac{2(\ell_2-l-1)\mathfrak{c}_{n-l}^+}{\tilde{\mathfrak{c}}_{-N}^-} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\mathcal{Q}_{12}} \\ 0 & k & 0 & \frac{\mathcal{I}_{21}}{\mathcal{Q}_{21}} \end{pmatrix}, & (4.86) \\
 B &= \begin{pmatrix} l-\ell_2 & 0 & \frac{\mathcal{I}_{12}}{\mathcal{Q}_{12}} & \frac{1}{\mathcal{Q}_{21}} \\ 0 & \ell_1-k & \frac{1}{\mathcal{Q}_{21}} & \frac{\mathcal{I}_{21}}{\mathcal{Q}_{12}} \\ (l-\ell_2)(N-\delta u) & (\ell_1-k)\ell_2\mathcal{I}_{12} & \frac{(N-\delta u-\ell_2)\mathcal{I}_{12}}{\mathcal{Q}_{12}} & \frac{N-\delta u-\ell_1-\ell_2}{\mathcal{Q}_{12}}\frac{\mathcal{I}_{12}\overline{\mathcal{Q}}_{12}}{\mathcal{Q}_{12}} \\ (\ell_2-l)(\ell_1\mathcal{I}_{21}) & (\ell_1-k)(\delta u-N) & \frac{\delta u-N+\ell_1}{\mathcal{Q}_{12}}\frac{\mathcal{I}_{12}\overline{\mathcal{Q}}_{12}+\ell_2}{\mathcal{Q}_{12}} & \frac{(\delta u-N+\ell_1)\mathcal{I}_{21}}{\mathcal{Q}_{12}} \end{pmatrix} \\
 &- 2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{l(1+n+k-\ell_1)(l-\ell_2)}{\tilde{\mathfrak{c}}_{-N}^-} & 0 & \frac{(l-\ell_2)(1+n+k-\ell_1)\mathcal{I}_{12}}{\tilde{\mathfrak{c}}_{-N}^-\mathcal{Q}_{12}} & \frac{(\ell_1-k)(1+N-n+l-\ell_2)}{\tilde{\mathfrak{c}}_{-N}^-\mathcal{Q}_{21}} \\ 0 & \frac{k(1+N-n+l-\ell_2)(k-\ell_1)}{\tilde{\mathfrak{c}}_{-N}^-} & \frac{(l-\ell_2)(1+n+k-\ell_1)}{\tilde{\mathfrak{c}}_{-N}^-\mathcal{Q}_{21}} & \frac{(\ell_1-k)(1+N-n+l-\ell_2)\mathcal{I}_{21}}{\tilde{\mathfrak{c}}_{-N}^-\mathcal{Q}_{12}} \end{pmatrix},
 \end{aligned}$$

where we defined

$$\tilde{\mathfrak{c}}_m^\pm = \delta u \pm \frac{\ell_1 + \ell_2}{2} - m + 1. \quad (4.87)$$

Notice the similarities between the matrices A, B and B^+, B^- . From this, it is now straightforward to extract $\mathcal{Y}_n^{k,l}$ by simple linear algebra

$$\mathcal{Y}_n^{k,l} = A^{-1} \begin{pmatrix} \frac{\mathcal{Q}_{32}}{\mathcal{Q}_{34}} \frac{\mathcal{Q}_{31}}{\mathcal{Q}_{34}} & 0 & 0 \\ \frac{\mathcal{Q}_{42}}{\mathcal{Q}_{43}} \frac{\mathcal{Q}_{41}}{\mathcal{Q}_{43}} & 0 & 0 \\ 0 & 0 & \frac{\mathcal{Q}_{32}}{\mathcal{Q}_{34}} \frac{\mathcal{Q}_{31}}{\mathcal{Q}_{34}} \\ 0 & 0 & \frac{\mathcal{Q}_{42}}{\mathcal{Q}_{43}} \frac{\mathcal{Q}_{41}}{\mathcal{Q}_{43}} \end{pmatrix} \left\{ \mathcal{X}_n^{k+1,l-1} B^+ + \mathcal{X}_n^{k-1,l+1} B^- + \mathcal{X}_n^{k,l} B \right\}. \quad (4.88)$$

Note that the final result for $\mathcal{Y}_n^{k,l}$ purely depends on the spectral parameters through their difference δu , and the representation parameters only appear in the combinations $\mathcal{Q}_{ij}, \mathcal{I}_{ij}$ (modulo perhaps the overall scalar factor, which, as usual, has to be determined separately). The rest of the formula is taken care of purely by combinatorial factors involving the integer bound state components.

4.7 Complete Solution of Case III

We will perform here a similar construction as done in the previous section, in order to solve Case III in terms of Case II. Let us first set few additional notations. We introduce the S-matrix at this level in the following way:

$$\mathbb{S}|k, l\rangle_i^{\text{III}} \equiv \sum_{m=0}^{k+l} \sum_{j=1}^6 \mathcal{Z}_{m;i}^{k,l;j} |m, k+l-m\rangle_j^{\text{III}}. \quad (4.89)$$

It is clear that one can repeat a very similar derivation as performed in (4.76) and (4.77), where, instead of \mathcal{X} , one has to think of having \mathcal{Y} (and indices running over the appropriate domains). This time, one again considers the action of $\Delta\mathbb{Q}_3^1, \Delta\mathbb{Q}_2^{\dagger 4}$. The result is now the following matrix equations:

$$\begin{pmatrix} (n-\ell_1)a_3 & 0 & b_3 & 0 & b_4 & 0 \\ (N-n-\ell_2)a_4 & b_4 & 0 & 0 & 0 & -b_3 \\ 0 & (n-\ell_1)a_3 & 0 & b_3 & (N-n-\ell_2)a_4 & 0 \\ 0 & 0 & (N-n-\ell_2)a_4 & b_4 & 0 & (n-\ell_1)a_3 \end{pmatrix} \mathcal{Z}_n^{k,l} = \mathcal{Y}_n^{k,l} \begin{pmatrix} (k-\ell_1)a_1 & 0 & b_1 & 0 & b_2 & 0 \\ (l-\ell_2)a_2 & b_2 & 0 & 0 & 0 & -b_1 \\ 0 & (k-\ell_1)a_1 & 0 & b_1 & (l-\ell_2)a_2 & 0 \\ 0 & 0 & (l-\ell_2)a_2 & b_2 & 0 & (k-\ell_1)a_1 \end{pmatrix}, \quad (4.90)$$

and

$$\begin{pmatrix} (n-\ell_1)c_3 & 0 & d_3 & 0 & d_4 & 0 \\ (N-n-\ell_2)c_4 & d_4 & 0 & 0 & 0 & -d_3 \\ 0 & (n-\ell_1)c_3 & 0 & d_3 & (N-n-\ell_2)c_4 & 0 \\ 0 & 0 & (N-n-\ell_2)c_4 & d_4 & 0 & (n-\ell_1)c_3 \end{pmatrix} \mathcal{Z}_n^{k,l} = \mathcal{Y}_n^{k,l} \begin{pmatrix} (k-\ell_1)c_1 & 0 & d_1 & 0 & d_2 & 0 \\ (l-\ell_2)c_2 & d_2 & 0 & 0 & 0 & -d_1 \\ 0 & (k-\ell_1)c_1 & 0 & d_1 & (l-\ell_2)c_2 & 0 \\ 0 & 0 & (l-\ell_2)c_2 & d_2 & 0 & (k-\ell_1)c_1 \end{pmatrix}, \quad (4.91)$$

where

$$\mathcal{Z}_n^{k,l} \equiv \begin{pmatrix} \mathcal{Z}_{n;1}^{k,l;1} & \mathcal{Z}_{n;2}^{k,l;1} & \mathcal{Z}_{n;3}^{k,l;1} & \mathcal{Z}_{n;4}^{k,l;1} & \mathcal{Z}_{n;5}^{k,l;1} & \mathcal{Z}_{n;6}^{k,l;1} \\ \mathcal{Z}_{n;1}^{k,l;2} & \mathcal{Z}_{n;2}^{k,l;2} & \mathcal{Z}_{n;3}^{k,l;2} & \mathcal{Z}_{n;4}^{k,l;2} & \mathcal{Z}_{n;5}^{k,l;2} & \mathcal{Z}_{n;6}^{k,l;2} \\ \mathcal{Z}_{n;1}^{k,l;3} & \mathcal{Z}_{n;2}^{k,l;3} & \mathcal{Z}_{n;3}^{k,l;3} & \mathcal{Z}_{n;4}^{k,l;3} & \mathcal{Z}_{n;5}^{k,l;3} & \mathcal{Z}_{n;6}^{k,l;3} \\ \mathcal{Z}_{n;1}^{k,l;4} & \mathcal{Z}_{n;2}^{k,l;4} & \mathcal{Z}_{n;3}^{k,l;4} & \mathcal{Z}_{n;4}^{k,l;4} & \mathcal{Z}_{n;5}^{k,l;4} & \mathcal{Z}_{n;6}^{k,l;4} \\ \mathcal{Z}_{n;1}^{k,l;5} & \mathcal{Z}_{n;2}^{k,l;5} & \mathcal{Z}_{n;3}^{k,l;5} & \mathcal{Z}_{n;4}^{k,l;5} & \mathcal{Z}_{n;5}^{k,l;5} & \mathcal{Z}_{n;6}^{k,l;5} \\ \mathcal{Z}_{n;1}^{k,l;6} & \mathcal{Z}_{n;2}^{k,l;6} & \mathcal{Z}_{n;3}^{k,l;6} & \mathcal{Z}_{n;4}^{k,l;6} & \mathcal{Z}_{n;5}^{k,l;6} & \mathcal{Z}_{n;6}^{k,l;6} \end{pmatrix}. \quad (4.92)$$

Once again, the relation with (4.6) is apparent.

It is readily checked that in this case these equations are not all independent. Hence, one similarly needs additional equations, very much like in the previous section in order to compute \mathcal{Y} . In that case, these additional equations were provided by Yangian generators. In this case we are more fortunate and do not need the Yangian, since one can consider the action of $\Delta\mathbb{Q}_4^2$ and $\Delta\mathbb{Q}_1^{\dagger 3}$. It is easy to check that, by repeating the above procedure using these additional symmetries, one arrives at the following matrix equations:

$$\begin{pmatrix} na_3 & 0 & b_3 & 0 & 0 & -b_4 \\ (N-n)a_4 & b_4 & 0 & 0 & b_3 & 0 \\ 0 & na_3 & 0 & b_3 & 0 & (N-n)a_4 \\ 0 & 0 & (N-n)a_4 & b_4 & -na_3 & 0 \end{pmatrix} \mathcal{Z}_n^{k,l} = \tilde{\mathcal{Y}}_n^{k,l} \begin{pmatrix} ka_1 & 0 & b_1 & 0 & 0 & -b_2 \\ la_2 & b_2 & 0 & 0 & b_1 & 0 \\ 0 & ka_1 & 0 & b_1 & 0 & la_2 \\ 0 & 0 & la_2 & b_2 & -ka_1 & 0 \end{pmatrix} \quad (4.93)$$

and

$$\begin{pmatrix} nc_3 & 0 & d_3 & 0 & 0 & -d_4 \\ (N-n)c_4 & d_4 & 0 & 0 & d_3 & 0 \\ 0 & nc_3 & 0 & d_3 & 0 & (N-n)c_4 \\ 0 & 0 & (N-n)c_4 & d_4 & -nc_3 & 0 \end{pmatrix} \mathcal{Z}_n^{k,l} = \tilde{\mathcal{Y}}_n^{k,l} \begin{pmatrix} kc_1 & 0 & d_1 & 0 & 0 & -d_2 \\ lc_2 & d_2 & 0 & 0 & d_1 & 0 \\ 0 & kc_1 & 0 & d_1 & 0 & lc_2 \\ 0 & 0 & lc_2 & d_2 & -kc_1 & 0 \end{pmatrix}, \quad (4.94)$$

where we have defined

$$\tilde{\mathcal{Y}}_n^{k,l} \equiv \begin{pmatrix} \mathcal{Y}_{n-1;1}^{k-1,l;1} & \mathcal{Y}_{n-1;2}^{k,l-1;1} & \mathcal{Y}_{n-1;3}^{k-1,l;1} & \mathcal{Y}_{n-1;4}^{k,l-1;1} \\ \mathcal{Y}_{n;1}^{k-1,l;2} & \mathcal{Y}_{n;2}^{k,l-1;2} & \mathcal{Y}_{n;3}^{k-1,l;2} & \mathcal{Y}_{n;4}^{k,l-1;2} \\ \mathcal{Y}_{n-1;1}^{k-1,l;3} & \mathcal{Y}_{n-1;2}^{k,l-1;3} & \mathcal{Y}_{n-1;3}^{k-1,l;3} & \mathcal{Y}_{n-1;4}^{k,l-1;3} \\ \mathcal{Y}_{n;1}^{k-1,l;4} & \mathcal{Y}_{n;2}^{k,l-1;4} & \mathcal{Y}_{n;3}^{k-1,l;4} & \mathcal{Y}_{n;4}^{k,l-1;4} \end{pmatrix}. \quad (4.95)$$

Combining all of the above equations is sufficient in order to solve for \mathcal{Z} . To be more precise, one can write the equation for $\mathcal{Z}_n^{k,l}$ in the following way:

$$\begin{pmatrix} (n-\ell_1)\mathcal{Q}_{43} & 0 & \mathcal{I}_{43} & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\mathcal{I}_{43} \\ 0 & (n-\ell_1)\mathcal{Q}_{43} & 0 & \mathcal{I}_{43} & 0 & 0 \\ -n\mathcal{Q}_{43} & 0 & -\mathcal{I}_{43} & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & -\mathcal{I}_{43} & 0 \\ 0 & -n\mathcal{Q}_{43} & 0 & -\mathcal{I}_{43} & 0 & 0 \end{pmatrix} \mathcal{Z}_n^{k,l} = \tilde{\mathcal{Y}}_n^{k,l} \begin{pmatrix} (\ell_1-k)\mathcal{Q}_{14} & 0 & \mathcal{I}_{41} & 0 & \mathcal{I}_{42} & 0 \\ (l-\ell_2)\mathcal{Q}_{42} & \mathcal{I}_{42} & 0 & 0 & 0 & -\mathcal{I}_{41} \\ 0 & (\ell_1-k)\mathcal{Q}_{14} & 0 & \mathcal{I}_{41} & (l_2-l)\mathcal{Q}_{42} & 0 \\ 0 & 0 & (l-\ell_2)\mathcal{Q}_{42} & \mathcal{I}_{42} & 0 & (\ell_1-k)\mathcal{Q}_{14} \\ k\mathcal{Q}_{14} & 0 & -\mathcal{I}_{41} & 0 & 0 & \mathcal{I}_{42} \\ -l\mathcal{Q}_{42} & -\mathcal{I}_{42} & 0 & 0 & -\mathcal{I}_{41} & 0 \\ 0 & k\mathcal{Q}_{14} & 0 & -\mathcal{I}_{41} & 0 & -l\mathcal{Q}_{42} \\ 0 & 0 & -l\mathcal{Q}_{42} & -\mathcal{I}_{42} & -k\mathcal{Q}_{14} & 0 \end{pmatrix}, \quad (4.96)$$

with

$$\tilde{\mathcal{Y}}_n^{k,l} \equiv \begin{pmatrix} \mathcal{Y}_{n;1}^{k,l;1} & \mathcal{Y}_{n;2}^{k,l;1} & \mathcal{Y}_{n;3}^{k,l;1} & \mathcal{Y}_{n;4}^{k,l;1} & 0 & 0 & 0 & 0 \\ \mathcal{Y}_{n;1}^{k,l;2} & \mathcal{Y}_{n;2}^{k,l;2} & \mathcal{Y}_{n;3}^{k,l;2} & \mathcal{Y}_{n;4}^{k,l;2} & 0 & 0 & 0 & 0 \\ \mathcal{Y}_{n;1}^{k,l;3} & \mathcal{Y}_{n;2}^{k,l;3} & \mathcal{Y}_{n;3}^{k,l;3} & \mathcal{Y}_{n;4}^{k,l;3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{Y}_{n-1;1}^{k-1,l;1} & \mathcal{Y}_{n-1;2}^{k,l-1;1} & \mathcal{Y}_{n-1;3}^{k-1,l;1} & \mathcal{Y}_{n-1;4}^{k,l-1;1} \\ 0 & 0 & 0 & 0 & \mathcal{Y}_{n;1}^{k-1,l;2} & \mathcal{Y}_{n;2}^{k,l-1;2} & \mathcal{Y}_{n;3}^{k-1,l;2} & \mathcal{Y}_{n;4}^{k,l-1;2} \\ 0 & 0 & 0 & 0 & \mathcal{Y}_{n-1;1}^{k-1,l;3} & \mathcal{Y}_{n-1;2}^{k,l-1;3} & \mathcal{Y}_{n-1;3}^{k-1,l;3} & \mathcal{Y}_{n-1;4}^{k,l-1;3} \end{pmatrix}. \quad (4.97)$$

The explicit matrix inversion gives

$$\mathcal{Z} = \begin{pmatrix} \frac{1}{\ell_1\mathcal{Q}_{34}} & \frac{1}{\ell_1\mathcal{Q}_{34}\mathcal{I}_{43}} & 0 & \frac{1}{\ell_1\mathcal{Q}_{34}} & \frac{1}{\ell_1\mathcal{Q}_{34}\mathcal{I}_{43}} & 0 \\ 0 & 0 & \frac{1}{\ell_1\mathcal{Q}_{34}} & 0 & 0 & \frac{1}{\ell_1\mathcal{Q}_{34}} \\ \frac{n}{\ell_1\mathcal{I}_{43}} & \frac{n-\ell_1}{\ell_1\mathcal{I}_{43}^2} & \frac{1}{\ell_1\mathcal{Q}_{34}\mathcal{I}_{43}^2} & \frac{n-\ell_1}{\ell_1\mathcal{I}_{43}} & \frac{n}{\ell_1\mathcal{I}_{43}^2} & \frac{1}{\ell_1\mathcal{Q}_{34}\mathcal{I}_{43}^2} \\ 0 & 0 & \frac{n}{\ell_1\mathcal{I}_{43}} & 0 & 0 & \frac{n-\ell_1}{\ell_1\mathcal{I}_{43}} \\ 0 & 0 & \frac{1}{\ell_1\mathcal{Q}_{43}\mathcal{I}_{43}} & 0 & -\frac{1}{\mathcal{I}_{43}} & \frac{1}{\ell_1\mathcal{Q}_{43}\mathcal{I}_{43}} \\ 0 & -\frac{1}{\mathcal{I}_{43}} & \frac{1}{\ell_1\mathcal{Q}_{34}\mathcal{I}_{43}} & 0 & 0 & \frac{1}{\ell_1\mathcal{Q}_{34}\mathcal{I}_{43}} \end{pmatrix} \tilde{\mathcal{Y}}_n^{k,l} \times \begin{pmatrix} (\ell_1-k)\mathcal{Q}_{14} & 0 & \mathcal{I}_{41} & 0 & \mathcal{I}_{42} & 0 \\ (l-\ell_2)\mathcal{Q}_{42} & \mathcal{I}_{42} & 0 & 0 & 0 & -\mathcal{I}_{41} \\ 0 & (\ell_1-k)\mathcal{Q}_{14} & 0 & \mathcal{I}_{41} & (l_2-l)\mathcal{Q}_{42} & 0 \\ 0 & 0 & (l-\ell_2)\mathcal{Q}_{42} & \mathcal{I}_{42} & 0 & (\ell_1-k)\mathcal{Q}_{14} \\ k\mathcal{Q}_{14} & 0 & -\mathcal{I}_{41} & 0 & 0 & \mathcal{I}_{42} \\ -l\mathcal{Q}_{42} & -\mathcal{I}_{42} & 0 & 0 & -\mathcal{I}_{41} & 0 \\ 0 & k\mathcal{Q}_{14} & 0 & -\mathcal{I}_{41} & 0 & -l\mathcal{Q}_{42} \\ 0 & 0 & -l\mathcal{Q}_{42} & -\mathcal{I}_{42} & -k\mathcal{Q}_{14} & 0 \end{pmatrix}. \quad (4.98)$$

It is now straightforward to do the matrix multiplication. This solves the final case. Once again, the dependence of the entries solely on the difference of the spectral parameters, and on the characteristic combinations of representation labels already observed in Case II, is a noticeable feature of the result.

4.8 Reduction and Comparison

Let us now compare our formulae with the known S-matrices. Here, one runs into potential difficulties. The formulae from the previous sections were derived for generic bound states, and one might wonder whether there could be obstructions for small bound states. A first problem arises when n is comparable to ℓ_1, ℓ_2 . A second problem is encountered for $n = 0, n = k + l$, since the basis of two-particle states in these two cases is lower-dimensional. One may wonder whether our formulae

$$\mathbb{S}|k, l\rangle^{\text{I}} = \sum_{n=0}^{k+l} \mathcal{X}_n^{k,l} |n, N - n\rangle^{\text{I}} \quad (4.99)$$

$$\mathbb{S}|k, l\rangle_i^{\text{II}} = \sum_{n=0}^{k+l} \sum_{j=1}^4 \mathcal{Y}_{n;i}^{k,l;j} |n, N - n\rangle_j^{\text{II}} \quad (4.100)$$

$$\mathbb{S}|k, l\rangle_i^{\text{III}} = \sum_{n=0}^{k+l} \sum_{j=1}^6 \mathcal{Z}_{n;i}^{k,l;j} |n, N - n\rangle_j^{\text{III}}, \quad (4.101)$$

with $N = k + l$ and \mathcal{Y}, \mathcal{Z} given by (4.88) and (4.98), remain valid also for these particular values.

It turns out that this is indeed the case. Let us deal with the first problem. One can see from (4.64) that, when $n > \ell_1$, precisely the unwanted S-matrix elements vanish, basically thanks to the vanishing of the correspondent coefficients $\mathcal{X}_n^{k,l}$.

Concerning the second potential problem, we notice that the issue arises only for Case II and III states. In Case II, the corresponding sum on the right hand side of (4.100) contains terms like

$$\mathcal{Y}_{0;i}^{k,l;4} |0, N\rangle_4^{\text{II}}. \quad (4.102)$$

But, as seen from (4.13), $|0, N\rangle_4^{\text{II}}$ is not well-defined (actually it is not part of our bound state representation). Hence, the S-matrix transition amplitudes toward these states, $\mathcal{Y}_{0;i}^{k,l;4}$, should vanish identically. We verified that this indeed turns out to be the case, which means that these states completely decouple.

More specifically, from (4.64) it can be shown that

$$\mathcal{X}_0^{k+1,l-1} = \frac{\ell_2 - l}{\delta u - \frac{\ell_1 - \ell_2}{2} - l + 1} \mathcal{X}_0^{k,l} \quad (4.103)$$

$$\mathcal{X}_0^{k-1,l+1} = \frac{\delta u - \frac{\ell_1 - \ell_2}{2} - l}{\ell_2 - l - 1} \mathcal{X}_0^{k,l}. \quad (4.104)$$

This means that in (4.88) one can pull out a factor $\mathcal{X}_0^{k,l}$. The remaining matrix part is straightforwardly seen to have zeroes for the states corresponding to the amplitudes $\mathcal{Y}_{0;i}^{k,l;4}$, for all

$i = 1, 2, 3, 4$ as it should indeed be the case. In other words, one can unambiguously write

$$\mathbb{S}|k, l\rangle_i^\Pi = \sum_{n=0}^{k+l} \sum_{j=1}^4 \mathcal{Y}_{n;i}^{k,l;j} |n, N-n\rangle_j^\Pi, \quad (4.105)$$

where $\mathcal{Y}_{n;i}^{k,l;j}$ is given by the complete 4×4 matrix from (4.88). The same is true for Case III states.

One can now compare our coefficients against the known S-matrices. Complete agreement is found with $\mathbb{S}^{AA}, \mathbb{S}^{AB}, \mathbb{S}^{BB}$ from [38, 40, 84]. We also checked several coefficients of the S-matrix $\mathbb{S}^{1\ell}$ from [63], and also in that case we find agreement with our results.

To make this comparison more explicit, let us list some explicit entries from the S-matrix. These entries can be directly compared against the coefficients from the known S-matrices like the fundamental S-matrix (4.1). The coefficients we list here will also be needed in later chapters.

The lowest entries of the Case II S-matrix are given by²

$$\mathcal{Y}_{k;1}^{k,0;1} = \frac{x_1^+ - x_2^+}{x_1^- - x_2^+} \sqrt{\frac{x_1^-}{x_1^+}} \left[1 - \frac{k}{\delta u + \frac{\ell_1 - \ell_2}{2}} \right] \mathcal{X}_k^{k,0}, \quad (4.106)$$

$$\mathcal{Y}_{k;2}^{k,0;2} = \frac{x_1^- - x_2^-}{x_1^- - x_2^+} \sqrt{\frac{x_2^+}{x_2^-}} \mathcal{X}_k^{k,0}, \quad (4.107)$$

$$\mathcal{Y}_{k;2}^{k,0;1} = \frac{x_2^- - x_2^+}{x_1^- - x_2^+} \sqrt{\frac{x_1^- x_2^+}{x_1^+ x_2^-}} \frac{\sqrt{\ell_1} \eta(p_1)}{\sqrt{\ell_2} \eta(p_2)} \frac{k - \ell_1}{\ell_1} \mathcal{X}_k^{k,0}, \quad (4.108)$$

$$\mathcal{Y}_{k;1}^{k,0;2} = \frac{x_1^+ - x_1^-}{x_1^- - x_2^+} \frac{\sqrt{\ell_2} \eta(p_2)}{\sqrt{\ell_1} \eta(p_1)} \mathcal{X}_k^{k,0}, \quad (4.109)$$

$$\mathcal{Y}_{k;1}^{k,0;4} = \frac{\sqrt{\ell_1 \ell_2} \eta(p_1) \eta(p_2)}{x_1^+ x_2^+ - 1} \frac{k}{i \ell_1} \mathcal{X}_k^{k,0}, \quad (4.110)$$

$$\mathcal{Y}_{k;4}^{k,0;4} = \sqrt{\frac{x_2^+}{x_2^-} \frac{x_1^+ x_2^- - 1}{x_1^+ x_2^+ - 1}} \mathcal{X}_k^{k,0}, \quad (4.111)$$

$$\mathcal{Y}_{k;4}^{k,0;1} = \frac{i}{\sqrt{\ell_1 \ell_2} \eta(p_1) \eta(p_2)} \sqrt{\frac{x_1^+ x_2^+}{x_1^- x_2^-}} \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)}{x_1^+ x_2^+ - 1} \mathcal{X}_k^{k,0}, \quad (4.112)$$

²We suppress the dependence on momenta in order to have a lighter notation. All functions appearing in this section have to be understood as $\mathcal{X} \equiv \mathcal{X}(p_1, p_2)$, $\mathcal{Y} \equiv \mathcal{Y}(p_1, p_2)$, $\mathcal{Z} \equiv \mathcal{Z}(p_1, p_2)$ (indices are omitted here for simplicity).

From the Case III S-matrix we will encounter

$$\mathcal{Z}_{k;1}^{k,0;1} = \left[1 - \frac{2ik}{g} \frac{x_1^+(x_2^- - x_1^-x_1^+x_2^+)}{(x_2^- - x_1^+)(1 - x_1^-x_1^+)(1 - x_1^+x_2^+)} \right] \frac{\mathcal{X}_k^{k,0}}{\mathcal{D}}, \quad (4.113)$$

$$\mathcal{Z}_{k;3}^{k,0;1} = \frac{2(x_1^- - x_1^+)(x_1^+)^2(x_2^- - x_2^+)}{g(x_1^+ - x_2^-)(1 - x_1^+x_1^-)(1 - x_1^+x_2^+)\eta(p_1)^2} \frac{\mathcal{X}_k^{k,0}}{\mathcal{D}}, \quad (4.114)$$

$$\mathcal{Z}_{k;1}^{k,0;3} = \frac{ik(\ell_1 - k)}{\ell_1} \frac{(x_2^- - x_2^+)\eta(p_1)^2}{(x_1^+ - x_2^-)(1 - x_1^+x_2^+)} \frac{\mathcal{X}_k^{k,0}}{\mathcal{D}}, \quad (4.115)$$

$$\mathcal{Z}_{k;3}^{k,0;3} = \left[\frac{(x_1^+ - x_2^+)(1 - x_2^-x_1^+)}{(x_1^+ - x_2^-)(1 - x_1^+x_2^+)} + \frac{2ik}{g} \frac{x_1^+(x_2^+ - x_1^-x_1^+x_2^-)}{(x_1^+ - x_2^-)(1 - x_1^-x_1^+)(1 - x_1^+x_2^+)} \right] \frac{\mathcal{X}_k^{k,0}}{\mathcal{D}} \quad (4.116)$$

and

$$\mathcal{Z}_{k;1}^{k,0;6} = \frac{ik\sqrt{\ell_2}\eta(p_1)\eta(p_2)}{\sqrt{\ell_1}} \frac{x_1^- - x_2^-}{(x_1^- - x_2^+)(1 - x_1^+x_2^+)} \sqrt{\frac{x_2^+}{x_2^-}} \mathcal{X}_k^{k,0}, \quad (4.117)$$

$$\mathcal{Z}_{k;3}^{k,0;6} = \sqrt{\frac{\ell_2}{\ell_1}} \frac{\eta(p_2)}{\eta(p_1)} \frac{(x_1^- - x_1^+)(x_2^-x_1^+ - 1)}{(x_1^- - x_2^+)(x_1^+x_2^+ - 1)} \sqrt{\frac{x_2^+}{x_2^-}} \mathcal{X}_k^{k,0}, \quad (4.118)$$

$$\mathcal{Z}_{k;6}^{k,0;6} = \frac{(x_1^- - x_2^-)(x_2^-x_1^+ - 1)x_2^+}{(x_1^- - x_2^+)(x_1^+x_2^+ - 1)x_2^-} \mathcal{X}_k^{k,0}, \quad (4.119)$$

$$\mathcal{Z}_{k;6}^{k,0;3} = \frac{(\ell_1 - k)\eta(p_1)}{\sqrt{\ell_1\ell_2}\eta(p_2)} \frac{(x_2^-x_1^+ - 1)(x_2^- - x_2^+)x_2^+}{(x_1^- - x_2^+)(x_1^+x_2^+ - 1)x_2^-} \sqrt{\frac{x_1^-}{x_1^+}} \mathcal{X}_k^{k,0}, \quad (4.120)$$

$$\mathcal{Z}_{k;6}^{k,0;1} = \frac{i}{\sqrt{\ell_1\ell_2}\eta(p_1)\eta(p_2)} \frac{(x_1^- - x_2^-)(x_1^- - x_1^+)(x_2^- - x_2^+)x_2^+}{(x_1^- - x_2^+)(x_1^+x_2^+ - 1)x_2^-} \sqrt{\frac{x_1^+}{x_1^-}} \mathcal{X}_k^{k,0}. \quad (4.121)$$

4.9 Summary

In this chapter we explicitly constructed S-matrix that intertwines two symmetric short representations of \mathfrak{h} by using Yangian symmetry.

Because of $\mathfrak{su}(2) \times \mathfrak{su}(2)$ invariance, when this S-matrix acts on such a (tensor-product) bound state representation space, it leaves five different subspaces invariant. Each of these subspaces is characterized by a specific assignment of $\mathfrak{su}(2) \times \mathfrak{su}(2)$ Dynkin labels, which are quantum numbers that are trivially conserved during the scattering. We found that two pairs of these subspaces are simply related to each other by exchanging the type of fermions. Therefore, we found only three non-equivalent cases. The S-matrix has the following block-diagonal form:

$$\mathbb{S} = \begin{pmatrix} \boxed{\mathcal{X}} & & & & \\ & \boxed{\mathcal{Y}} & & & \\ & & \boxed{\mathcal{Z}} & & \\ & 0 & & \boxed{\mathcal{Y}} & \\ & & & & \boxed{\mathcal{X}} \end{pmatrix}. \quad (4.122)$$

The outer blocks scatter states from Case I (4.12)

$$\mathcal{X} : |k, l\rangle^{\text{I}} \mapsto \sum_{m=0}^{k+l} \mathcal{X}_m^{k,l} |m, k+l-m\rangle^{\text{I}}, \quad (4.123)$$

where $\mathcal{X}_m^{k,l}$ is given by equation (4.64). The blocks \mathcal{Y} describe the scattering of states from Case II (4.13)

$$\mathcal{Y} : |k, l\rangle_j^{\text{II}} \mapsto \sum_{m=0}^{k+l} \sum_{j=1}^4 \mathcal{Y}_{m;i}^{k,l;j} |m, k+l-m\rangle_j^{\text{II}}. \quad (4.124)$$

These S-matrix elements are given in equation (4.88). Finally, the middle block deals with the third case (4.14)

$$\mathcal{Z} : |k, l\rangle_j^{\text{III}} \mapsto \sum_{m=0}^{k+l} \sum_{j=1}^6 \mathcal{Z}_{m;i}^{k,l;j} |m, k+l-m\rangle_j^{\text{III}}, \quad (4.125)$$

where $\mathcal{Z}_{m;i}^{k,l;j}$ can be found in (4.98).

This S-matrix is canonically normalized, namely, it leaves the vacuum state (4.24) exactly invariant. The full $\text{AdS}_5 \times \text{S}^5$ string bound state S-matrix is obtained by taking two copies of the above derived S-matrix and multiplying *each one of them* with the phase factor [84]

$$S_0(p_1, p_2) = \left(\frac{x_1^-}{x_1^+} \right)^{\frac{\ell_2}{2}} \left(\frac{x_2^+}{x_2^-} \right)^{\frac{\ell_1}{2}} \sigma(x_1, x_2) \sqrt{G(\ell_2 - \ell_1) G(\ell_2 + \ell_1)} \prod_{q=1}^{\ell_1-1} G(\ell_2 - \ell_1 + 2q), \quad (4.126)$$

where G has been given in (4.3). Concretely the complete S-matrix is given by

$$\mathbb{S}_{\text{Full}}(p_1, p_2) = S_0(p_1, p_2)^2 \mathbb{S}(p_1, p_2) \otimes \mathbb{S}(p_1, p_2), \quad (4.127)$$

which for explicit bound state numbers ℓ_1, ℓ_2 is a $(16\ell_1\ell_2)^2$ by $(16\ell_1\ell_2)^2$ dimensional matrix.

The Classical r-Matrix

For many integrable models the semiclassical limit of the S-matrix is described by a special object, called classical r -matrix. This matrix is a universal structure in the sense that it can be defined purely in algebraic terms and hence it is representation independent. Its possible forms have been classified for simple Lie algebras [137] and have also been studied for certain superalgebras [138–140]. However, those results do not apply to \mathfrak{h} .

Consider a generic Yangian evaluation representation and rescale $u \rightarrow u/\hbar$. It is readily seen that the coproduct of the Yangian generators (2.33) becomes of the form

$$\Delta \hat{\mathbb{J}}^A = \hat{\mathbb{J}}^A \otimes \mathbb{1} + \mathbb{1} \otimes \hat{\mathbb{J}}^A + \hbar f_{BC}^A \mathbb{J}^B \otimes \mathbb{J}^C. \quad (5.1)$$

Obviously, this induces an \hbar dependence in the S-matrix, and one generically finds that for small \hbar

$$\mathbb{S} = \mathbb{1} + \hbar r + \mathcal{O}(\hbar^2). \quad (5.2)$$

The matrix r is called the classical r -matrix. For the Yangian of \mathfrak{h} the situation is non-generic since u is actually fixed in terms of the underlying representation. The conventions used here are such that a rescaling by \hbar is not needed, instead one can define a classical limit by identifying $\hbar = g^{-1}$.

Generically, the classical r -matrix for Yangians in the evaluation representation can be given purely in terms of algebra generators

$$r = \frac{K_{AB} \mathbb{J}^A \otimes \mathbb{J}^B}{u_1 - u_2}, \quad (5.3)$$

where K_{AB} is the Killing form. It satisfies the classical Yang-Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \quad (5.4)$$

However, since the Killing form K_{AB} of \mathfrak{h} is zero, this formula is not applicable. Nevertheless, the classical limit of the S-matrix discussed in the previous chapter was studied in [141] and shortly after a proposal of the classical r -matrix for the $\text{AdS}_5 \times \text{S}^5$ superstring was put forward in [100]. A different proposal closer to (5.3) was made in [101]. In this chapter we will discuss both proposals and show that the second indeed agrees with the (semi-)classical limit of the bound state S-matrices.

5.1 The near plane-wave limit

As was discussed in Section 3.2.2, in order to describe bound states with momentum p and bound state number ℓ it is convenient to introduce parameters x^\pm such that

$$\frac{x^+}{x^-} = e^{ip}, \quad x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i\ell}{g}, \quad (5.5)$$

where $g = \sqrt{\lambda}/2\pi$ is related to the 't Hooft coupling λ . A convenient parameterization of x^\pm that can be used to study the large coupling limit $g \rightarrow \infty$ was introduced in [142].

$$x_i^\pm = x_i \left(\sqrt{1 - \frac{(\ell_i/g)^2}{(x_i - \frac{1}{x_i})^2}} \pm \frac{i\ell_i/g}{x_i - \frac{1}{x_i}} \right). \quad (5.6)$$

In this parametrization, the Hamiltonian is given by

$$H = \ell \frac{x^2 + 1}{x^2 - 1} \quad (5.7)$$

and the momentum is related to the parameter x by

$$\sin \frac{p}{2} = \frac{\ell}{g} \frac{x}{x^2 - 1}. \quad (5.8)$$

Notice that the energy does not depend explicitly on the coupling g .

The large coupling limit corresponds to the semiclassical limit of spinning strings [14] or to the near plane-wave limit [143]. In other words, we can indeed identify the inverse coupling $1/g$ with the \hbar parameter discussed above. In the remainder of the chapter we will study the classical r -matrix by taking the $g \rightarrow \infty$ limit of the S-matrix (4.17). To this end we will need the expansion of the parameters a, b, c, d (3.13) describing the bound state representation. We denote the near-classical limit of these parameters as $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$:

$$\begin{aligned} \mathfrak{a} &= \frac{x}{\sqrt{x^2 - 1}}, & \mathfrak{b} &= -\frac{1}{\sqrt{x^2 - 1}}, \\ \mathfrak{c} &= -\frac{1}{\sqrt{x^2 - 1}}, & \mathfrak{d} &= \frac{x}{\sqrt{x^2 - 1}}. \end{aligned} \quad (5.9)$$

The square root factors are coming from expanding η (3.14). The other parameter that appears in the S-matrices is the Yangian evaluation parameter u (3.76) and this reduces to

$$u = -\frac{ig}{2} \left(x + \frac{1}{x} \right) + \frac{i\ell^2}{4g} \frac{x + \frac{1}{x}}{\left(x - \frac{1}{x} \right)^2} + \mathcal{O}(g^{-3}). \quad (5.10)$$

Finally the braiding factor \mathbb{U} is related to the momentum p and its semiclassical expansion is easily found to be

$$\sqrt{\frac{x^+}{x^-}} = 1 + \frac{i\ell}{g} \frac{x}{x^2 - 1}. \quad (5.11)$$

5.2 The Moriyama-Torrielli proposal

The first proposal for the classical r-matrix has been made in [100] and is given by

$$\begin{aligned} r = \sum_{n=0}^{\infty} & \left[(\mathbb{L}_b^a)_n \otimes (\tilde{\mathbb{L}}_a^b)_{-n-1} - (\mathbb{L}_b^a)_{-n-1} \otimes (\tilde{\mathbb{L}}_a^b)_n - \right. \\ & (\mathbb{R}_\beta^\alpha)_n \otimes (\tilde{\mathbb{R}}_\alpha^\beta)_{-n-1} + (\mathbb{R}_\beta^\alpha)_{-n-1} \otimes (\tilde{\mathbb{R}}_\alpha^\beta)_n + \\ & \left. + (\mathbb{Q}_\alpha^a)_n \otimes (\mathbb{Q}_a^{\dagger\alpha})_{-n-1} - (\mathbb{Q}_\alpha^a)_{-n-1} \otimes (\mathbb{Q}_a^{\dagger\alpha})_n + \mathbb{H}_n \otimes \mathbb{B}_{-n-1} + \mathbb{B}_n \otimes \mathbb{H}_{-n-1} \right]. \end{aligned} \quad (5.12)$$

where the bosonic generators are given by

$$\begin{aligned} (\mathbb{L}_b^a)_n &= \frac{x^{n+1} - x^{-n-1}}{x - \frac{1}{x}} \mathbb{L}_b^a, & (\tilde{\mathbb{L}}_b^a)_n &= \frac{x^{n-1} - x^{1-n}}{x - \frac{1}{x}} \mathbb{L}_b^a, \\ (\mathbb{R}_\beta^\alpha)_n &= \frac{x^{1+n} - x^{-1-n}}{x - \frac{1}{x}} \mathbb{R}_\beta^\alpha, & (\tilde{\mathbb{R}}_\beta^\alpha)_n &= \frac{x^{n-1} - x^{1-n}}{x - \frac{1}{x}} \mathbb{R}_\beta^\alpha, \\ \mathbb{H}_n = \tilde{\mathbb{H}}_n &= \frac{x^{n+1} + x^{-n-1}}{x + \frac{1}{x}} \mathbb{H}, & \mathbb{B}_n = \tilde{\mathbb{B}}_n &= \frac{x^n - x^{-n}}{2} \frac{x + \frac{1}{x}}{x - \frac{1}{x}} \mathbb{B}. \end{aligned} \quad (5.13)$$

The operator \mathbb{B} is the extra generator that makes $\mathfrak{su}(2|2)$ into $\mathfrak{u}(2|2)$ and is, in the operator language given by (3.45)

$$\mathbb{B} = \frac{1}{2(ad + bc)} \left(w_a \frac{\partial}{\partial w_a} - \theta_\alpha \frac{\partial}{\partial \theta_\alpha} \right). \quad (5.14)$$

The supersymmetry generators are given by

$$\begin{aligned} (\mathbb{Q}_\alpha^a)_n &= (\tilde{\mathbb{Q}}_\alpha^a)_n = \mathbb{Q}_\alpha^a (x^n \Pi_b + x^{-n} \Pi_f), \\ (\mathbb{Q}_a^{\dagger\alpha})_n &= (\tilde{\mathbb{Q}}_a^{\dagger\alpha})_n = \mathbb{Q}_a^{\dagger\alpha} (x^{-n} \Pi_b + x^n \Pi_f). \end{aligned} \quad (5.15)$$

In the above expression Π_b and Π_f are the projectors on the bosonic and fermionic subspace, respectively.

The dependence of the generators on the parameter x is quite different from the standard evaluation representation, especially because of the presence of the bosonic and fermionic projectors.

One can check that this classical r -matrix matches with the semiclassical limit of the fundamental S-matrix. Nevertheless, when one considers bound state S-matrices disagreement is found. This shows that this proposal for classical r -matrix is not universal.

5.3 The Beisert-Spill proposal

The other conjecture for the classical r -matrix was put forward in [101]. This is again done by introducing the extra generator \mathbb{B} from (3.45). This proposal for the classical r -matrix was made in terms of algebra generators in the evaluation representation and its form is close to the standard classical r -matrix for Yangians (5.3).

The r -matrix is given in terms of algebra generators and evaluation parameters u_1, u_2 . Consider the following two-site operator

$$\mathcal{T}_{12} = 2 \left(\mathbb{R}_\beta^\alpha \otimes \mathbb{R}_\alpha^\beta - \mathbb{L}_b^a \otimes \mathbb{L}_a^b + \mathbb{Q}_a^{\dagger\alpha} \otimes \mathbb{Q}_\alpha^a - \mathbb{Q}_\alpha^a \otimes \mathbb{Q}_a^{\dagger\alpha} \right). \quad (5.16)$$

In terms of the operator \mathbb{B} , the proposed classical r -matrix is [101]

$$r_{12} = \frac{g}{2} \left[\frac{\mathcal{T}_{12} - \mathbb{B} \otimes \mathbb{H} - \mathbb{H} \otimes \mathbb{B}}{u_1 - u_2} - \frac{\mathbb{B} \otimes \mathbb{H}}{u_2} + \frac{\mathbb{H} \otimes \mathbb{B}}{u_1} + \left(\frac{1}{u_1} - \frac{1}{u_2} \right) \mathbb{H} \otimes \mathbb{H} \right]. \quad (5.17)$$

All the operators and u in the above expression are understood in the strict classical limit, i.e. the lowest order terms in (5.9), (5.10). The last term is proportional to the identity operator and is related to the phase factor of the S-matrix. It was shown in [101] that r satisfies the classical Yang-Baxter equation.

Via (3.36) it is straightforward to put r into differential operator form since it is completely defined in terms of the algebra generators and central elements. Upon taking the near plane-wave limit discussed above we can then compare the action of this operator to the semiclassical limit of the bound state S-matrix.

For completeness, we give here the operator form of \mathcal{T}_{12} . The operator \mathcal{T}_{12} is composed of two operators acting in different spaces, whose superspace variables are again denoted by w_a, θ_α and v_a, ϑ_α respectively. Writing it out is straightforward:

$$\begin{aligned} \mathcal{T}_{12} = & (-2w_b v_a + w_a v_b) \frac{\partial^2}{\partial w_a \partial v_b} + (2\theta_\beta \vartheta_\alpha - \theta_\alpha \vartheta_\beta) \frac{\partial^2}{\partial \theta_\alpha \partial \vartheta_\beta} + \\ & 2(\mathfrak{a}_1 \mathfrak{d}_2 - \mathfrak{b}_2 \mathfrak{c}_1) v_a \theta_\alpha \frac{\partial^2}{\partial w_a \partial \vartheta_\alpha} + 2(\mathfrak{a}_2 \mathfrak{d}_1 - \mathfrak{b}_1 \mathfrak{c}_2) w_a \vartheta_\alpha \frac{\partial^2}{\partial v_a \partial \theta_\alpha} + \\ & 2(\mathfrak{a}_2 \mathfrak{c}_1 - \mathfrak{b}_1 \mathfrak{d}_2) \theta_\alpha \vartheta_\beta \epsilon_{ab} \epsilon^{\alpha\beta} \frac{\partial^2}{\partial w_a \partial v_b} + 2(\mathfrak{a}_1 \mathfrak{c}_2 - \mathfrak{b}_2 \mathfrak{d}_1) w_a v_b \epsilon^{ab} \epsilon_{\alpha\beta} \frac{\partial^2}{\partial \theta_\alpha \partial \vartheta_\beta}. \end{aligned} \quad (5.18)$$

The coefficients $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ are the classical limits of a, b, c, d defined above (5.9).

5.4 The semi-classical limit of the S-matrix

We will now concentrate on the plane-wave limit of the bound state S-matrices. In this limit it should agree with the universal classical r -matrix. There are two parts to the S-matrix, the matrix part and the overall scalar factor. We will start with a discussion of the latter.

The dressing phase

By making use of fusion techniques, the scalar factor of the bound state S-matrix scattering bound states of length ℓ_1, ℓ_2 respectively was found in [84]. It was shown that this factor respects crossing symmetry. Recall that, if one defines

$$G(n) := \frac{u_1 - u_2 + \frac{n}{2}}{u_1 - u_2 - \frac{n}{2}}, \quad (5.19)$$

the explicit form of the scalar factor is given by (4.126)

$$S_0^{\ell_1 \ell_2}(p_1, p_2) = \left(\frac{x_1^-}{x_1^+} \right)^{\frac{\ell_2}{2}} \left(\frac{x_2^+}{x_2^-} \right)^{\frac{\ell_1}{2}} \sigma(x_1, x_2) \sqrt{G(\ell_2 - \ell_1) G(\ell_1 + \ell_2)} \prod_{k=1}^{\ell_1 - 1} G(\ell_2 - \ell_1 + 2k),$$

where $\sigma(x_2, x_1)$ is the dressing factor [44]. For comparison with the classical r -matrix this has to be evaluated to order $\mathcal{O}(g^{-1})$, which will then be combined with the matrix part later on. First of all, the functions $G(n)$ and the factors proportional to the momenta are easily expanded around $g \rightarrow \infty$ by using (5.6). We find

$$G(n) = 1 + \frac{2in}{g} \frac{x_1 x_2}{(x_1 - x_2)(x_1 x_2 - 1)} + \mathcal{O}(g^{-2}). \quad (5.20)$$

To examine the dressing factor $\sigma(x_1, x_2)$, we first introduce the conserved charges

$$\begin{aligned} q_n(x_i) &\equiv \frac{i}{n-1} \left(\frac{1}{(x_i^+)^{n-1}} - \frac{1}{(x_i^-)^{n-1}} \right) \\ &= \frac{2\ell_i}{g} \frac{x_i^{2-n}}{x_i^2 - 1} + \mathcal{O}(g^{-2}). \end{aligned} \quad (5.21)$$

The dressing phase θ is related to the conserved charges as follows

$$\sigma(x_1, x_2) = e^{\frac{i}{2}\theta(x_1, x_2)}, \quad (5.22)$$

where

$$\theta_{12} = g \sum_{r=2}^{\infty} \sum_{n=0}^{\infty} c_{r,r+1+2n} (q_r(x_1) q_{r+1+2n}(x_2) - q_r(x_2) q_{r+1+2n}(x_1)), \quad (5.23)$$

with [142]

$$c_{r,s} = \delta_{r+1,s} - g^{-1} \frac{4}{\pi} \frac{(r-1)(s-1)}{(r+s-2)(s-r)} + \mathcal{O}(g^{-2}). \quad (5.24)$$

Since in the near plane-wave limit $q_n \sim g^{-1}$, we see that if we work to order $\mathcal{O}(g^{-1})$, it suffices to take $c_{r,s} = \delta_{r+1,s}$. Hence, the dressing phase reduces to

$$\begin{aligned}
\theta_{12} &= g \sum_{r=2}^{\infty} \sum_{n=0}^{\infty} \delta_{n,0} (q_r(x_1) q_{r+1+2n}(x_2) - q_r(x_2) q_{r+1+2n}(x_1)) + \mathcal{O}(g^{-2}) \\
&= g \sum_{r=2}^{\infty} (q_r(x_1) q_{r+1}(x_2) - q_r(x_2) q_{r+1}(x_1)) + \mathcal{O}(g^{-2}) \\
&= \frac{4\ell_1\ell_2}{g} \frac{x_1^2 x_2^2 (x_1 - x_2)}{(x_1^2 - 1)(x_2^2 - 1)} \sum_{r=2}^{\infty} \left(\frac{1}{x_1 x_2} \right)^{r+1} + \mathcal{O}(g^{-2}) \\
&= \frac{4\ell_1\ell_2}{g} \frac{(x_1 - x_2)}{(x_1^2 - 1)(x_1 x_2 - 1)(x_2^2 - 1)} + \mathcal{O}(g^{-2}). \tag{5.25}
\end{aligned}$$

This gives

$$\sigma(x_1, x_2) = 1 + \frac{2i\ell_1\ell_2}{g} \frac{(x_1 - x_2)}{(x_1^2 - 1)(x_1 x_2 - 1)(x_2^2 - 1)} + \mathcal{O}(g^{-2}). \tag{5.26}$$

From this expression it is easy to see that, at least to first order, the dressing phases of bound states indeed respect fusion.

The remainder of $S_0^{\ell_1\ell_2}$ is easily found to give

$$\begin{aligned}
\left(\frac{x_1^-}{x_1^+} \right)^{\frac{\ell_2}{2}} \left(\frac{x_2^+}{x_2^-} \right)^{\frac{\ell_1}{2}} \sqrt{G(\ell_2 - \ell_1)G(\ell_1 + \ell_2)} \prod_{k=1}^{\ell_1-1} G(\ell_2 - \ell_1 + 2k) = \\
1 - \frac{i\ell_1\ell_2}{g} \left(\frac{x_1}{x_1^2 - 1} - \frac{2x_1 x_2}{(x_1 - x_2)(x_1 x_2 - 1)} - \frac{x_2}{x_2^2 - 1} \right) + \mathcal{O}(g^{-2}). \tag{5.27}
\end{aligned}$$

Combining this with the dressing phase, we obtain in the near plane-wave limit

$$S_0^{\ell_1\ell_2}(p_1, p_2) = 1 + \frac{i\ell_1\ell_2}{g} \frac{(x_1 x_2 - 1)(x_1^2 + x_2^2)}{(x_1^2 - 1)(x_1 - x_2)(x_2^2 - 1)} + \mathcal{O}(g^{-2}). \tag{5.28}$$

Matrix Part

Let us now turn to the matrix part. We will first study the S-matrix \mathcal{R} (4.64) from Case I in detail.

When looking at formula (4.64), one can see that, besides expanding the factor \mathcal{D} , one needs to expand the remaining expression, depending only on the difference δu of the spectral parameters, for large values of δu . The terms relevant to the classical limit of (4.64) are given

by the following expansion:

$$\begin{aligned} \mathcal{R}_n^{k,l} \sim (1 + \mathcal{D}_{cl}) & \frac{\left(1 - \frac{1}{\delta u} \sum_{p=1}^{k+l} \left(\frac{\ell_1 + \ell_2}{2} - p\right)\right) \prod_{i=1}^n (\ell_1 - i) \prod_{i=1}^{k+l-n} (\ell_2 - i)}{\prod_{p=1}^k (\ell_1 - p) \prod_{p=1}^l (\ell_2 - p)} \times \\ & \times \sum_{m=0}^k \left\{ \left(1 + \frac{1}{\delta u} \sum_{p=1}^m \left(\frac{\ell_1 - \ell_2}{2} + 1 - p\right) + \frac{1}{\delta u} \sum_{p=1-m}^{l-n} \left(\frac{\ell_2 - \ell_1}{2} + 1 - p\right)\right) \times \right. \\ & \left. \times \delta u^{2m-k-n} \binom{k}{k-m} \binom{l}{n-m} \prod_{p=1}^{k-m} \mathfrak{d}_{\frac{k-p+2}{2}} \prod_{p=1}^{n-m} \tilde{\mathfrak{d}}_{\frac{k+l-m-p+2}{2}} \right\}, \end{aligned} \quad (5.29)$$

where \mathcal{D}_{cl} denotes the first order in $1/g$ of \mathcal{D} . Here, we have used the fact that the binomials enforce $l \geq n - m$, in order to obtain the power of δu^{2m-k-n} . Let us start by considering non-diagonal amplitudes, namely, n different from k (cfr. (4.63)). In order to do that, let us first reduce the above formula for the case $n \geq k$. In this case, the leading piece in the above expression is given by the term in the sum with $m = k$ (the binomials are in this case non-zero, since, from (4.63), one has $l \geq n - k$). The amplitude tends to

$$\mathcal{R}_n^{k,l} \sim \frac{1}{\delta u^{n-k}} \frac{\prod_{i=1}^n (\ell_1 - i) \prod_{i=1}^{k+l-n} (\ell_2 - i)}{\prod_{p=1}^k (\ell_1 - p) \prod_{p=1}^l (\ell_2 - p)} \binom{l}{n-k} \prod_{p=1}^{n-k} \tilde{\mathfrak{d}}_{\frac{l-p+2}{2}}. \quad (5.30)$$

As one can see, in the non-diagonal case only one of these amplitudes actually contributes to the classical limit (corresponding to the order $1/g$ of the scattering matrix). Namely, only the transition from a state characterized by quantum number k to one with corresponding quantum number $n = k + 1$ has the right order, the other ones being suppressed by higher powers of δu . In this situation, the classical amplitudes reads

$$\mathcal{R}_{k+1}^{k,l} \sim \frac{1}{\delta u} l(\ell_1 - k - 1). \quad (5.31)$$

Next, let us consider $k \geq n$. In this case the binomials force the leading piece in the sum to be the one with $m = n$. This reads (quite symmetrically w.r.t the previous case)

$$\mathcal{R}_n^{k,l} \sim \frac{1}{\delta u^{k-n}} \frac{\prod_{i=1}^n (\ell_1 - i) \prod_{i=1}^{k+l-n} (\ell_2 - i)}{\prod_{p=1}^k (\ell_1 - p) \prod_{p=1}^l (\ell_2 - p)} \binom{k}{k-n} \prod_{p=1}^{k-n} \mathfrak{d}_{\frac{k-p+2}{2}}. \quad (5.32)$$

Analogously, only one of the non-diagonal terms has the right falloff to be able to contribute to the classical r-matrix, namely the amplitude for quantum numbers k to $n = k - 1$. The contribution is given by

$$\mathcal{R}_{k-1}^{k,l} \sim \frac{1}{\delta u} k(\ell_2 - l - 1). \quad (5.33)$$

The diagonal part, for $n = k$, is slightly more complicated. The leading term can be obtained by restricting to $k = n$ either of the two formulas (5.30) or (5.32), and is easily seen to be equal

to 1. The S-matrix tends in fact to the identity in the strict classical limit. The next to leading term of order $1/\delta u$ contributes to the classical r-matrix, and can be straightforwardly obtained from (5.29) as

$$\mathcal{R}_k^{k,l} \sim 1 + D_{cl} + \frac{1}{\delta u} \left[\sum_{p=1}^{k+l} \left(\frac{\ell_1 + \ell_2}{2} - p \right) + \sum_{p=1}^k \left(\frac{\ell_1 - \ell_2}{2} + 1 - p \right) + \sum_{p=1-k}^{l-k} \left(\frac{\ell_2 - \ell_1}{2} + 1 - p \right) \right].$$

Having now determined the semi-classical limit of the Case I S-matrix, one can easily produce the semi-classical limit of the Case II S-matrix (4.88). By expanding the matrices A, B, B^\pm to order g^{-1} one again finds that the only terms $\mathcal{Y}_{n;j}^{k,l;i}$ that survive are those with $k = n$ and $k = n \pm 1$. A similar discussion also holds for the Case III S-matrix.

5.5 Comparison in the near plane-wave limit

We will now compare the classical r -matrix (5.17) against the semi-classical limit of the bound state S-matrix. Let us first look at the dressing phase. To this end, we recall that the bound state S-matrices $\mathbb{S}^{\ell_1 \ell_2}$ are canonically normalized by setting

$$\mathbb{S}_{can}^{\ell_1 \ell_2} w_1^{\ell_1} v_1^{\ell_2} = w_1^{\ell_1} v_1^{\ell_2}. \quad (5.34)$$

For the fully dressed S-matrix we therefore obtain

$$\mathbb{S}^{\ell_1 \ell_2} w_1^{\ell_1} v_1^{\ell_2} = S_0^{\ell_1 \ell_2} w_1^{\ell_1} v_1^{\ell_2}, \quad (5.35)$$

where S_0 is the scalar factor given by (4.126). On the other hand, assuming that the classical r -matrix is universal, we can easily compute its action on the state $w_1^{\ell_1} v_1^{\ell_2}$. One finds

$$(1 + g^{-1}r) w_1^{\ell_1} v_1^{\ell_2} = S_0^{\ell_1 \ell_2} (x_1, x_2) w_1^{\ell_1} v_1^{\ell_2}. \quad (5.36)$$

This means that the phase factor (4.126) derived in [84] is indeed compatible with r .

The matrix structure is now easily compared by acting with r as an operator on states $|k, l\rangle^I, |k, l\rangle_i^{II}, |k, l\rangle_i^{III}$ and comparing the coefficients of the resulting states against the S-matrix elements. Doing this leads to perfect agreement.

As a curiosity, we note that the classical r-matrix actually can be used to describe the S-matrices up to second order (apart from the overall factor)

$$\mathbb{S} \sim 1 + \frac{r}{g} + \frac{r^2}{2g^2} + \mathcal{O}(g^{-3}). \quad (5.37)$$

It is easily checked that this exponential pattern breaks down at third order.

5.6 Summary

In this chapter we compared the classical limit of the bound state S-matrix against two different algebraic expressions for the classical r -matrix. We find that the universal r -matrix (5.12) from [100] agrees with the semi-classical limit of the fundamental S-matrix but this agreement breaks down for higher bound state numbers. However, the universal r -matrix (5.17) put forward in [101] correctly describes the semi-classical limit of all bound state S-matrices. Moreover, it even captures the matrix structure at one order higher.

Chapter 6

Universal Blocks

In chapter 2 the notion of a quasi-triangular Hopf algebra was introduced. These Hopf algebras admit an R-matrix, which is a purely algebraic object intertwining the coproduct and the opposite coproduct (for the relevant definitions we remind to section 2.3). When evaluated in explicit representations the R-matrix gives rise to the S-matrix in that representation.

Of course, in a concrete representation one can compute the S-matrix without knowing the universal R-matrix. This is what was done in chapter 4, where bound state S-matrices were computed. One might then wonder if this S-matrix (or parts thereof) has an algebraic origin. The existence of a universal R-matrix would be interesting from a mathematical point of view and useful for computing the S-matrix in representations for which a direct derivation may be cumbersome.

In this chapter we will explore this universality. The bound state representations we have discussed in section 3.4 contain representations of subalgebras of \mathfrak{h} . We can identify two such subspaces, namely a $\mathfrak{gl}(1|1)$ and an $\mathfrak{su}(2)$. The $\mathfrak{su}(2)$ is particularly interesting since it describes Case I states, which provided the starting point for our construction in chapter 4. The (double) Yangian of both subalgebras admits a universal R-matrix and one can ask whether the restriction of the S-matrix to these states corresponds to this universal R-matrix. We will see that this is indeed the case.

6.1 The $\mathfrak{su}(2)$ subspace

The first subspace is given by states that span Case I. We remind that the $\mathfrak{psu}(2|2)$ algebra has two (“bosonic” and “fermionic”, according to the indices they transform) $\mathfrak{su}(2)$ subalgebras, with

generators \mathbb{L}_b^a and \mathbb{R}_β^α , respectively. The first ones satisfy the following commutation relations:

$$\begin{aligned} [\mathbb{L}_a^b, \mathbb{J}_c] &= \delta_c^b \mathbb{J}_a - \frac{1}{2} \delta_a^b \mathbb{J}_c, \\ [\mathbb{L}_a^b, \mathbb{J}^c] &= -\delta_a^c \mathbb{J}^b + \frac{1}{2} \delta_a^b \mathbb{J}^c. \end{aligned} \quad (6.1)$$

The states

$$\theta_3 w_1^{\ell-k-1} w_2^k, \quad (6.2)$$

form a natural representation on which the “bosonic” $\mathfrak{su}(2)$ subalgebra of \mathbb{L}_b^a ’s acts. They form an $\ell - 1$ dimensional representation. Obviously any vector $|k, l\rangle^I$ from Case I (4.12) originates from the tensor product of two such states. It is easy to see that the coproducts of the Yangian generators on these states coincide with the truncation to the $\mathfrak{su}(2)$ generators of the general expressions (3.67). Furthermore, the Case I S-matrix satisfies the Yang-Baxter equation by itself, and it is of difference form. This means that such S-matrix should naturally come from the universal R-matrix of the $\mathfrak{su}(2)$ Yangian double [116].

Drinfeld II for $\mathfrak{su}(2)$

In [116], the universal R-matrix for Yangian doubles has been constructed using Drinfeld’s second realization of the Yangian. The discussion there can straightforwardly be applied to the bound state representations in the superspace formalism.

The map between the first and the second realization becomes

$$\begin{aligned} \kappa_0 &= 2\mathbb{L}_2^2, & \xi_0^+ &= \mathbb{L}_2^1, & \xi_0^- &= \mathbb{L}_1^2, \\ \kappa_1 &= 2\hat{\mathbb{L}}_2^2 - v, & \xi_1^+ &= \hat{\mathbb{L}}_2^1 - w, & \xi_1^- &= \hat{\mathbb{L}}_1^2 - z, \end{aligned} \quad (6.3)$$

where

$$v = \frac{1}{2}(\{\mathbb{L}_1^2, \mathbb{L}_2^1\} - (\mathbb{L}_2^2)^2), \quad w = -\frac{1}{4}\{\mathbb{L}_2^1, \mathbb{L}_2^2\}, \quad z = -\frac{1}{4}\{\mathbb{L}_1^2, \mathbb{L}_2^2\}. \quad (6.4)$$

The operators \mathbb{L}_b^a are realized as in (3.36). The higher level generators are given by

$$\begin{aligned} \xi_n^- &= \xi_0^-(u + \frac{\kappa_0 - 1}{2})^n, \\ \xi_n^+ &= \xi_0^+(u + \frac{\kappa_0 + 1}{2})^n, \\ \kappa_n &= \xi_0^+ \xi_n^- - \xi_0^- \xi_n^+. \end{aligned} \quad (6.5)$$

The parameter u corresponds to (3.77). The generators (6.5) coincide with those obtained in [116] for generic highest-weight representations of $Y(\mathfrak{su}(2))$. It is easy to check that these

generators satisfy the correct relations

$$\begin{aligned}
[\kappa_m, \kappa_n] &= 0, & [\xi_m^+, \xi_n^-] &= \kappa_{n+m}, \\
[\kappa_0, \xi_m^+] &= 2\xi_m^+, & [\kappa_0, \xi_m^-] &= -2\xi_m^-, \\
[\kappa_{m+1}, \xi_n^+] - [\kappa_m, \xi_{n+1}^+] &= \{\kappa_m, \xi_n^+\}, \\
[\kappa_{m+1}, \xi_n^-] - [\kappa_m, \xi_{n+1}^-] &= -\{\kappa_m, \xi_n^-\}, \\
[\xi_{m+1}^+, \xi_n^+] - [\xi_m^+, \xi_{n+1}^+] &= \{\xi_m^+, \xi_n^+\}, \\
[\xi_{m+1}^-, \xi_n^-] - [\xi_m^-, \xi_{n+1}^-] &= -\{\xi_m^-, \xi_n^-\}
\end{aligned} \tag{6.6}$$

that define Drinfeld's second realization of $Y(\mathfrak{su}(2))$.

The universal R-matrix

We will now proceed to compute the universal R-matrix for the $\mathfrak{su}(2)$ block of our bound state S-matrix, following [116]. The derivation is split up into three parts, corresponding to the factorization

$$R = R_+ R_0 R_-, \tag{6.7}$$

R_+ and R_- being “root” factors, while R_0 is a purely diagonal “Cartan” factor. The different terms are

$$R_+ = \prod_{n \geq 0}^{\rightarrow} \exp(-\xi_n^+ \otimes \xi_{-n-1}^-), \tag{6.8}$$

$$R_- = \prod_{n \geq 0}^{\leftarrow} \exp(-\xi_n^- \otimes \xi_{-n-1}^+), \tag{6.9}$$

$$R_0 = \prod_{n \geq 0} \exp \left\{ \text{Res}_{u=v} \left[\frac{d}{du} (\log H^+(u)) \otimes \log H^-(v + 2n + 1) \right] \right\}. \tag{6.10}$$

One has defined

$$\text{Res}_{u=v} (A(u) \otimes B(v)) = \sum_k a_k \otimes b_{-k-1} \tag{6.11}$$

for $A(u) = \sum_k a_k u^{-k-1}$ and $B(u) = \sum_k b_k u^{-k-1}$, and the so-called Drinfeld's currents are given by

$$\begin{aligned}
E^\pm(u) &= \pm \sum_{\substack{n \geq 0 \\ n < 0}} \xi_n^\pm u^{-n-1}, & F^\pm(u) &= \pm \sum_{\substack{n \geq 0 \\ n < 0}} \xi_n^\mp u^{-n-1} \\
H^\pm(u) &= 1 \pm \sum_{\substack{n \geq 0 \\ n < 0}} \kappa_n u^{-n-1}.
\end{aligned} \tag{6.12}$$

The arrows on the products indicate the ordering one has to follow in the multiplication, and are a consequence of the normal ordering prescription for the root factors in the universal R-matrix [116]. For the generic bound state representations which we have described above, the ordering will be essential to get the correct result. To keep notation concise we introduce

$$\langle A, B \rangle \langle C, D \rangle = \theta_3 w_1^A w_2^B \vartheta_3 v_1^C v_2^D, \quad \langle A, B \rangle = \theta_3 w_1^A w_2^B. \quad (6.13)$$

In this notation we have for the state $|k, l\rangle^I$, $A = \ell_1 - k - 1, B = k, C = \ell_2 - l - 1, D = l$. Let us first compute how R_- acts on an arbitrary Case I state. We find

$$\prod_{n \geq 0}^{\leftarrow} \exp[-\xi_n^- \otimes \xi_{-1-n}^+] |k, l\rangle = \sum_m A_m |k - m, l + m\rangle. \quad (6.14)$$

The term A_m is built up out of m copies of $-\xi^- \otimes \xi^+$ acting on the state $\langle A, B \rangle \langle C, D \rangle$, which is made of an A number of w_1 's, a B number of w_2 's in the first space, and analogously C and D for v_1, v_2 in the second space. In view of (6.14), we find that such terms can come from different exponentials, i.e. with different n 's, or from the same exponential. One first needs to know how the product of m ξ^+ 's acts on the state $\langle A, B \rangle$. We conveniently define

$$\begin{aligned} c_i &= u_1 - \frac{A - B + 1}{2} - i, & d_i &= u_2 - \frac{C - D - 1}{2} + i, \\ \tilde{c}_i &= u_2 - \frac{C - D + 1}{2} - i, & \tilde{d}_i &= u_1 - \frac{A - B - 1}{2} + i, \end{aligned} \quad (6.15)$$

and

$$\delta u = u_1 - u_2. \quad (6.16)$$

In general one has

$$\begin{aligned} \xi_{n_m}^- \dots \xi_{n_2}^- \xi_{n_1}^- \langle A, B \rangle &= \xi_{n_m}^- \dots \xi^- \left(u + \frac{h-1}{2}\right)^{n_2} \xi^- \left(u + \frac{h-1}{2}\right)^{n_1} \langle A, B \rangle \\ &= \xi_{n_m}^- \dots \xi^- \left(u + \frac{h-1}{2}\right)^{n_2} \xi^- (c_0)^{n_1} \langle A, B \rangle \\ &= B (c_0)^{n_1} \xi_{n_m}^- \dots \xi^- \left(u + \frac{h-1}{2}\right)^{n_2} \langle A + 1, B - 1 \rangle \\ &= B(B-1) (c_0)^{n_1} (c_1)^{n_2} \xi_{n_m}^- \dots \xi_{n_3}^- \langle A + 2, B - 2 \rangle \\ &= \frac{B!}{(B-m)!} c_0^{n_1} \dots c_{m-1}^{n_m} \langle A + m, B - m \rangle. \end{aligned} \quad (6.17)$$

Similar expressions hold for ξ_n^+ acting on $\langle C, D \rangle$, but with d_i instead of c_i , and producing the state $\langle C - m, D + m \rangle$. When we consider terms like this coming from the ordered exponential (6.14), we always have that $n_i \geq n_{i-1}$. In case $n_i = n_{i+1}$, we also pick up a combinatorial factor

coming from the series of the exponential. Putting all of this together, we find

$$A_m = (-)^m \frac{B!}{(B-m)!} \frac{C!}{(C-m)!} \left\{ \sum_{n_1 \leq \dots \leq n_m} \frac{1}{N(\{n_1, \dots, n_m\})} \frac{c_0^{n_1}}{d_0^{n_1+1}} \cdots \frac{c_{m-1}^{n_m}}{d_{m-1}^{n_m+1}} \right\},$$

$$N(\{n_1, \dots, n_m\}) = \frac{1}{\text{ord}S(\{n_1, \dots, n_m\})}. \quad (6.18)$$

N is a combinatorial factor which is defined as the inverse of the order of the permutation group of the set $\{n_1, \dots, n_m\}$. For example, $N(\{1, 1, 2\}) = \frac{1}{2}$ and $N(\{1, 1, 1, 2, 3, 3, 4, 5\}) = \frac{1}{3!} \frac{1}{2!} = \frac{1}{12}$. By using the fact that $c_i = c_{i+1} + 1, d_i = d_{i+1} - 1$, one can evaluate this sum explicitly and find

$$A_m(A, B, C, D) = m! \binom{B}{m} \binom{C}{m} \prod_{i=0}^{m-1} \frac{1}{c_0 - d_0 - i - m + 1}, \quad (6.19)$$

where we have indicated the dependence on the parameters A, B, C, D of the state we are acting on. As one can easily see using (6.15), the resulting expression is manifestly of difference form, i.e. it only depends on δu .

A similar consideration works for R_+ . One has

$$\prod_{n \geq 0}^{\rightarrow} \exp[-\xi_n^+ \otimes \xi_{-1-n}^-] |k, l\rangle = \sum_m B_m |k+m, l-m\rangle. \quad (6.20)$$

where

$$B_m(A, B, C, D) = m! \binom{A}{m} \binom{D}{m} \prod_{i=0}^{m-1} \frac{1}{\tilde{d}_0 - \tilde{c}_0 - i + m - 1}. \quad (6.21)$$

Finally, we turn to the Cartan part. First, we work out

$$\kappa_n \langle A, B \rangle = \left\{ (A+1)B \left[u - \frac{A-B+1}{2} \right]^n - (B+1)A \left[u - \frac{A-B-1}{2} \right]^n \right\} \langle A, B \rangle.$$

We then recall the definition of H_{\pm} from (6.12). From the explicit realization (6.22) it follows that

$$H_+(t) \langle A, B \rangle = H_-(t) \langle A, B \rangle$$

$$= \left\{ 1 - \frac{(A+1)B}{u-t-\frac{1}{2}(A-B+1)} + \frac{A(B+1)}{u-t-\frac{1}{2}(AB-1)} \right\} \langle A, B \rangle. \quad (6.22)$$

Defining $K_{\pm} = \log H_{\pm}$, the Cartan part of the universal R-matrix can be written as

$$R_0 = \prod_{n \geq 0} \exp \left[\text{Res}_{t=x} \left(\frac{d}{dt} K_+(t) \otimes K_-(x+2n+1) \right) \right], \quad (6.23)$$

where the residue is defined in (6.11). We have to find the suitable series representations corresponding to $\frac{d}{dt}K_+(t)$ and $K_-(x + 2n + 1)$. With an appropriate choice of domains for the variables t and x , one can write in particular

$$\frac{d}{dt}K_+(t) = \sum_{m \geq 1} \{\alpha_1^m + \alpha_2^m - \alpha_3^m - \alpha_4^m\} t^{-m-1}, \quad (6.24)$$

$$K_-(x + 2n + 1) = K_-(0) + \sum_{m \geq 1} \{\beta_1^{-m} + \beta_2^{-m} - \beta_3^{-m} - \beta_4^{-m}\} \frac{x^m}{m}, \quad (6.25)$$

where

$$\begin{aligned} \alpha_1 &= u_1 + \frac{1}{2}(A + B + 1), & \alpha_2 &= u_1 - \frac{1}{2}(A + B + 1), \\ \alpha_3 &= u_1 - \frac{1}{2}(A - B + 1), & \alpha_4 &= u_1 - \frac{1}{2}(A - B - 1), \end{aligned} \quad (6.26)$$

and

$$\begin{aligned} \beta_1 &= u_2 - 2n + \frac{1}{2}(D - C - 1), & \beta_2 &= u_2 - 2n + \frac{1}{2}(D - C - 3), \\ \beta_3 &= u_2 - 2n + \frac{1}{2}(D + C - 1), & \beta_4 &= u_2 - 2n - \frac{1}{2}(D + C + 3), \end{aligned} \quad (6.27)$$

This leads to

$$\begin{aligned} R_0 \langle A, B \rangle \langle C, D \rangle &= \frac{2^{1-2\delta u} \pi \Gamma\left(\frac{2\delta u + A + B + C - D + 2}{2}\right) \Gamma\left(\frac{2\delta u + B - A + C + D + 2}{2}\right)}{\Gamma\left(\frac{2\delta u + A + B - C - D + 2}{4}\right) \Gamma\left(\frac{\delta u - A + B + C - D + 2}{2}\right) \Gamma\left(\frac{2\delta u + A + B + C + D + 4}{4}\right)} \times \\ &\quad \times \frac{\Gamma\left(\frac{2\delta u - A + B - C - D}{2}\right) \Gamma\left(\frac{2\delta u - A - B + C - D}{2}\right)}{\Gamma\left(\frac{2\delta u - A - B + C + D + 2}{4}\right) \Gamma\left(\frac{\delta u - A + B + C - D}{2}\right) \Gamma\left(\frac{2\delta u - A - B - C - D}{4}\right)} \langle A, B \rangle \langle C, D \rangle \\ &\equiv \mathcal{H}(A, B, C, D) \langle A, B \rangle \langle C, D \rangle. \end{aligned} \quad (6.28)$$

We are now ready to put things together and evaluate the action of the universal R-matrix of $\mathfrak{su}(2)$ on Case I states. We obtain

$$\begin{aligned} R|k, l\rangle &= \sum_{m=0}^{\min(B, C)} \sum_{n=0}^{\min(A, D) + m} B_n(A + m, B - m, C - m, D + m) \\ &\quad \times \mathcal{H}(A + m, B - m, C - m, D + m) A_m(A, B, C, D) |k - m + n, l + m - n\rangle, \end{aligned} \quad (6.29)$$

where

$$\begin{aligned} A &= \ell_1 - k - 1, & B &= k, \\ C &= \ell_2 - l - 1, & D &= l, \end{aligned} \quad (6.30)$$

and the various factors are given by formulas (6.14), (6.20) and (6.28). It is now easy to convert the above expression into

$$R|k, l\rangle = \sum_{n=0}^{k+l} R_n |n, k + l - n\rangle. \quad (6.31)$$

In order to find the amplitudes R_n , we proceed as follows. Taking into account the presence of binomial factors in the expressions for A_m and B_n , which naturally truncate the sum when m, n lie outside the correct intervals, we can extend the summation indices to run from $-\infty$ to ∞ . In this way, manipulations of the above sums are easier, and one ends up with

$$\begin{aligned} R_n &= \sum_{m=-n+k}^{\infty} A_m(\ell_1 - k - 1, k, \ell_2 - l - 1, l) \\ &\quad \times \mathcal{H}(\ell_1 - k - 1 + m, k - m, \ell_2 - l - 1 - m, l + m) \\ &\quad \times B_{n-k+m}(\ell_1 - k - 1 + m, k - m, \ell_2 - l - 1 - m, l + m). \end{aligned} \quad (6.32)$$

We have checked that this coincides with the r.h.s. of (4.66) for a large selection of choices of the integer parameters, keeping δu arbitrary, when taking into account the proper normalization. We have in fact, with the notations of [88],

$$\mathcal{D} \frac{\Gamma\left(\frac{1}{4}(2 + \ell_1 - \ell_2 + 2\delta u)\right) \Gamma\left(\frac{1}{4}(2 + \ell_2 - \ell_1 + 2\delta u)\right)}{\Gamma\left(\frac{1}{4}(4 - \ell_1 - \ell_2 + 2\delta u)\right) \Gamma\left(\frac{1}{4}(\ell_1 + \ell_2 + 2\delta u)\right)} R_n = \mathcal{X}_n^{k,l}. \quad (6.33)$$

The ratio of gamma functions appearing in the above formula is the (inverse of the) so-called “character” of the universal R-matrix in evaluation representations [116], namely its action on states of highest-weight $\lambda_i = l_i - 1$.

6.2 Universal R-matrix for $\mathfrak{gl}(1|1)$

The other subspace we will consider is obtained by restricting the bound states to having bosonic and fermionic indices of only one respective type. For definiteness, we will take the bosonic index to be 1 and the fermionic index to be 3. There are four copies of this subspace, corresponding to the four different choices of these indices we can make. The embedding of this subspace in the full bound state representation is spanned by the vectors

$$\{|0, 0\rangle_1^{\text{III}}, |0, 0\rangle_1^{\text{II}}, |0, 0\rangle_2^{\text{II}}, |0, 0\rangle^{\text{I}}\}. \quad (6.34)$$

As one can see, this subspace takes particular states from all three Cases listed earlier, yet being closed under the action of the S-matrix. This means that the S-matrix for this subsector corresponds to a block-diagonal 4×4 matrix. Its form can be readily found from the explicit expressions in section 4.8. Putting this together, one obtains

$$\mathbb{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-i\frac{p_2}{2}} \frac{x_1^+ - x_2^+}{x_1^+ - x_2^-} & \frac{\sqrt{\ell_1}\eta(p_1)}{\sqrt{\ell_2}\eta(p_2)} \frac{x_2^+ - x_2^-}{x_1^+ - x_2^-} & 0 \\ 0 & \frac{e^{i\frac{p_1}{2}}}{e^{i\frac{p_2}{2}}} \frac{\sqrt{\ell_2}\eta(p_2)}{\sqrt{\ell_1}\eta(p_1)} \frac{x_1^+ - x_1^-}{x_1^+ - x_2^-} & e^{i\frac{p_1}{2}} \frac{x_1^- - x_2^-}{x_1^+ - x_2^-} & 0 \\ 0 & 0 & 0 & \frac{x_1^- - x_2^+}{x_1^+ - x_2^-} \frac{e^{i\frac{p_1}{2}}}{e^{i\frac{p_2}{2}}} \end{pmatrix}. \quad (6.35)$$

We remark that, taken in the fundamental representation, and suitably un-twisted in order to eliminate the braiding factors coming from the nontrivial coproduct [40, 125, 126], this matrix coincides with the S-matrix of [144]. It is readily checked that this matrix satisfies the Yang-Baxter equation by itself, therefore it is natural to ask whether it is the representation of a known (Yangian) universal R-matrix.

Drinfeld II

The algebra transforming the states inside these sectors is an $\mathfrak{sl}(1|1)$. As it is known, this type of superalgebras (with a degenerate Cartan matrix) do not admit a universal R-matrix, therefore we will introduce an extra Cartan generator [145] and study the Yangian of the algebra $gl(1|1)$ instead¹. Let us start with the canonical derivation, and adapt the representation later in order to exactly match with our S-matrix. We will follow [116, 117]. The super Yangian double $DY(gl(1|1))$ is the Hopf algebra generated by the elements ξ_n^+ , ξ_n^- , $\kappa_{1;n}$, $\kappa_{2;n}$, with n an integer number, satisfying (Drinfeld's second realization)

$$\begin{aligned}
[\kappa_{i;m}, \kappa_{j;n}] &= 0, \\
[\kappa_{2;m}, \xi_n^+] &= [\kappa_{2;m}, \xi_n^-] = 0, \\
[\kappa_{1;0}, \xi_n^+] &= -2\xi_n^+, \quad [\kappa_{1;0}, \xi_n^-] = 2\xi_n^-, \\
[\kappa_{1;m+1}, \xi_n^+] - [\kappa_{1;m}, \xi_{n+1}^+] + \{\kappa_{1;m}, \xi_n^+\} &= 0, \\
[\kappa_{1;m+1}, \xi_n^-] - [\kappa_{1;m}, \xi_{n+1}^-] - \{\kappa_{1;m}, \xi_n^-\} &= 0, \\
\{\xi_m^+, \xi_n^+\} &= \{\xi_m^-, \xi_n^-\} = 0, \\
\{\xi_m^+, \xi_n^-\} &= -\kappa_{2;m+n}.
\end{aligned} \tag{6.36}$$

Drinfeld's currents are given by

$$E^\pm(t) = \pm \sum_{\substack{n \geq 0 \\ n < 0}} \xi_n^\pm t^{-n-1}, \quad F^\pm(t) = \pm \sum_{\substack{n \geq 0 \\ n < 0}} \xi_n^\mp t^{-n-1}, \tag{6.37}$$

$$H^\pm(t) = 1 \pm \sum_{\substack{n \geq 0 \\ n < 0}} \kappa_{1;n} t^{-n-1}, \quad K^\pm(t) = 1 \pm \sum_{\substack{n \geq 0 \\ n < 0}} \kappa_{2;n} t^{-n-1}. \tag{6.38}$$

One can show that the following bound state representation, acting on monomials made of a generic bosonic state v and a generic fermionic state θ , satisfies all the defining relations of the second realization (6.36):

$$\begin{aligned}
\xi_n^+ &= \lambda^n \mathbb{Q}_3^1, & \xi_n^- &= \lambda^n \mathbb{Q}_1^{\dagger 3}, \\
\kappa_{1;n} &= 2(\lambda + \ell - 1)^n (\mathbb{L}_1^1 - \mathbb{R}_3^3), & \kappa_{2;n} &= -\lambda^n \left(\frac{1}{2} \mathbb{H} + \mathbb{L}_1^1 + \mathbb{R}_3^3 \right).
\end{aligned} \tag{6.39}$$

¹For the purposes of the universal R-matrix, it will not make any difference to consider real forms of the algebras when needed.

As usual, we denote by ℓ the number of components of the bound state. At this stage, λ is a generic spectral parameter and we leave the parameterization of $\mathbb{Q}, \mathbb{Q}^\dagger$ unspecified. One can see that the coproducts of the Yangian generators does not truncate nicely as in the $\mathfrak{su}(2)$ case. Because of this reason, we do not expect λ to agree with u . We will later specify the value λ has to take in order to match with the bound state S-matrix in these subsectors.

Universal R-matrix

The universal R -matrix reads

$$\mathcal{R} = \mathcal{R}_+ \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_-, \quad (6.40)$$

where

$$\mathcal{R}_+ = \prod_{n \geq 0}^{\rightarrow} \exp(-\xi_n^+ \otimes \xi_{-n-1}^-), \quad (6.41)$$

$$\mathcal{R}_- = \prod_{n \geq 0}^{\leftarrow} \exp(\xi_n^- \otimes \xi_{-n-1}^+), \quad (6.42)$$

$$\mathcal{R}_1 = \prod_{n \geq 0} \exp \left\{ \text{Res}_{t=z} \left[(-1) \frac{d}{dt} (\log H^+(t)) \otimes \ln K^-(z + 2n + 1) \right] \right\}, \quad (6.43)$$

$$\mathcal{R}_2 = \prod_{n \geq 0} \exp \left\{ \text{Res}_{t=z} \left[(-1) \frac{d}{dt} (\log K^+(t)) \otimes \ln H^-(z + 2n + 1) \right] \right\}, \quad (6.44)$$

and again

$$\text{Res}_{t=z} (A(t) \otimes B(z)) = \sum_k a_k \otimes b_{-k-1} \quad (6.45)$$

for $A(t) = \sum_k a_k t^{-k-1}$, $B(z) = \sum_k b_k z^{-k-1}$. We first compute

$$\mathcal{R}_- = \prod_{n \geq 0}^{\leftarrow} \exp[\xi_n^- \otimes \xi_{-n-1}^+] \quad (6.46)$$

in our representation (6.39). Because of the fermionic nature of the operators $\xi_n^- \otimes \xi_{-n-1}^+$, the above expression simplifies to

$$\begin{aligned} \mathcal{R}_- &= 1 + \sum_{n \geq 0} \xi_n^- \otimes \xi_{-n-1}^+ \\ &= 1 + \sum_{n \geq 0} \frac{\lambda_1^n}{\lambda_2^{n+1}} \xi_0^- \otimes \xi_0^+ \\ &= 1 - \frac{\xi^- \otimes \xi^+}{\delta \lambda} \end{aligned} \quad (6.47)$$

Considering that this term will act non-trivially only on states with a fermion in the first space, we easily obtain

$$\mathcal{R}_- = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{a_2 d_1 \ell_2}{\delta \lambda} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (6.48)$$

We have defined

$$\delta \lambda = \lambda_1 - \lambda_2. \quad (6.49)$$

Similarly, one finds

$$\begin{aligned} \mathcal{R}_+ &= 1 - \sum_{n \geq 0} \xi_n^+ \otimes \xi_{-n-1}^- \\ &= 1 + \frac{\xi_0^+ \otimes \xi_0^-}{\delta \lambda}, \end{aligned} \quad (6.50)$$

which, written as a matrix, takes the form

$$\mathcal{R}_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{a_1 d_2 \ell_1}{\delta \lambda} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (6.51)$$

Let us now turn to the Cartan part. For this, we first need to compute the currents (6.38). They are found to be

$$H^\pm = 1 - \frac{\kappa_{1;0}}{1 + \lambda - \ell - t}, \quad (6.52)$$

$$K^\pm = 1 - \frac{\kappa_{2;0}}{\lambda - t}, \quad (6.53)$$

where we used the fact that both $\kappa_{1;0}$ and $\kappa_{2;0}$ are diagonal operators. In appropriate domains of convergence of the series one then has in particular

$$-\frac{d}{dt} \log H^+ = \sum_{m=1}^{\infty} \{(\lambda + \ell - 1)^m - (\lambda + \ell - 1 - \kappa_{1;0})^m\} t^{-m-1} \quad (6.54)$$

and

$$\begin{aligned} \log K^-(z + 2n + 1) &= \log K^-(2n + 1) + \\ &+ \sum_{m=1}^{\infty} \left\{ \frac{1}{(\lambda - 1 - 2n)^m} - \frac{1}{(\lambda - 1 - 2n - \kappa_{2;0})^m} \right\} \frac{z^m}{m}. \end{aligned} \quad (6.55)$$

Straightforwardly computing the residue and performing the sum yields, in matrix form,

$$\mathcal{R}_1 = \frac{\Gamma\left(\frac{\delta\lambda+\ell_1}{2}\right) \Gamma\left(\frac{\delta\lambda-a_2d_2\ell_2}{2}\right)}{\Gamma\left(\frac{\delta\lambda}{2}\right) \Gamma\left(\frac{\delta\lambda+\ell_1-a_2d_2\ell_2}{2}\right)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\delta\lambda-a_2d_2\ell_2}{\delta\lambda} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{\delta\lambda-a_2d_2\ell_2}{\delta\lambda} \end{pmatrix}. \quad (6.56)$$

One can perform an analogous derivation for \mathcal{R}_2 and find

$$\mathcal{R}_2 = \frac{\Gamma\left(\frac{\delta\lambda+a_1d_1\ell_1+2}{2}\right) \Gamma\left(\frac{\delta\lambda-\ell_2+2}{2}\right)}{\Gamma\left(\frac{\delta\lambda+2}{2}\right) \Gamma\left(\frac{\delta\lambda+a_1d_1\ell_1-\ell_2+2}{2}\right)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\delta\lambda}{\delta\lambda+a_1d_1\ell_1} & 0 \\ 0 & 0 & 0 & \frac{\delta\lambda}{\delta\lambda+a_1d_1\ell_1} \end{pmatrix}. \quad (6.57)$$

Multiplying everything out finally gives us the universal R-matrix in our bound state representation:

$$\mathcal{R} = A \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \frac{a_2d_2\ell_2}{\delta\lambda+a_1d_1\ell_1} & \frac{a_1d_2\ell_1}{\delta\lambda+a_1d_1\ell_1} & 0 \\ 0 & \frac{a_2d_1\ell_2}{\delta\lambda+a_1d_1\ell_1} & \frac{\delta\lambda}{\delta\lambda+a_1d_1\ell_1} & 0 \\ 0 & 0 & 0 & \frac{\delta\lambda-a_2d_2\ell_2}{\delta\lambda+a_1d_1\ell_1} \end{pmatrix}, \quad (6.58)$$

where

$$A = \frac{\Gamma\left(\frac{\delta\lambda+\ell_1}{2}\right) \Gamma\left(\frac{\delta\lambda+a_1d_1\ell_1+2}{2}\right) \Gamma\left(\frac{\delta\lambda-\ell_2+2}{2}\right) \Gamma\left(\frac{\delta\lambda-a_2d_2\ell_2}{2}\right)}{\Gamma\left(\frac{\delta\lambda}{2}\right) \Gamma\left(\frac{\delta\lambda+2}{2}\right) \Gamma\left(\frac{\delta\lambda+a_1d_1\ell_1-\ell_2+2}{2}\right) \Gamma\left(\frac{\delta\lambda+\ell_1-a_2d_2\ell_2}{2}\right)}. \quad (6.59)$$

For $a_i = d_i = \ell_i = 1$ this reduces to the formula in [117],

$$\mathcal{R} \propto \mathbb{1} + \frac{P}{\delta\lambda}, \quad (6.60)$$

where P is the graded permutation matrix.

But we can also take a, d to be the representation labels of the supercharges in the centrally extended $\mathfrak{psu}(2|2)$ superalgebra, i.e.

$$a = \sqrt{\frac{g}{2\ell}}\eta, \quad d = \sqrt{\frac{g}{2\ell}} \frac{x^+ - x^-}{i\eta}. \quad (6.61)$$

This corresponds to considering the generators ξ^\pm as the restriction to this subsector of the two supercharges \mathbb{Q}_1^3 and $\mathbb{Q}_3^{\dagger 1}$. It is now readily seen that by choosing λ to be $\frac{g}{2i}x^-$, we can exactly reproduce² the 4×4 block (6.35) from (6.58), after we properly normalize it and introduce

²This is similar to the observation in [144] for the the case of the fundamental representation.

the appropriate braiding factors. To normalize, we simply divide the formula coming from the universal R-matrix by A (6.59). To introduce the braiding factors, we need to twist it by [84]

$$U_2^{-1}(p_1) \mathcal{R} U_1(p_2),$$

with $U(p) = \text{diag}(1, e^{-ip/2})$.

One can also restrict the supercharges \mathbb{Q}_2^4 and $\mathbb{Q}_4^{\dagger 2}$ to this sector and repeat the procedure. Remarkably, in order to match with (6.35), one has to choose $\lambda = \frac{ig}{2x^-}$. The correct braiding factors can be incorporated by means of the inverse of the above mentioned twist [84].

A similar argument can finally be seen to hold for all the other subsectors corresponding to different choices of the fixed bosonic and fermionic indices.

While it is likely that in the full universal R-matrix (where one is supposed to have at once all generators of $\mathfrak{psu}(2|2)$) some kind of “average” of the two situations will occur³, we have shown here that the S-matrix in these subspaces can be “effectively” described by the universal R-matrix of $DY(\mathfrak{gl}(1|1))$ taken in (two inequivalent choices of) evaluation representations.

6.3 Summary

In this chapter we explored some universal structures underlying the bound state S-matrix that was derived in chapter 4. We found two representations that are contained in the bound state representation of \mathfrak{h} , namely a $\mathfrak{su}(2)$ and a $\mathfrak{gl}(1|1)$. For these algebras the universal R-matrix is known [116, 117] and one finds that the restriction of the bound state S-matrix to these subspaces indeed agrees with these universal R-matrices.

³In the fundamental representation, this is exemplified by some of the formulas in [146].

The Coordinate Bethe Ansatz

The spectrum in integrable models can often be computed exactly by a technique called the Bethe ansatz. It was first used in the context of the one-dimensional XXX Heisenberg model in 1931 [45]. In this approach one makes an explicit plane-wave type ansatz for the eigenvectors of the Hamiltonian. On these eigenstates one can then impose periodic boundary conditions, which result in a quantization condition of the particle momenta. From this ansatz one computes the exact eigenvalues of the Hamiltonian for the system. This version of the Bethe ansatz is commonly called the coordinate Bethe ansatz.

The coordinate Bethe ansatz procedure for theories with multiple species of particles was first solved for a system with a repulsive δ -interaction [147]. As we will see, the fact that there are different types of particles in the model results in a certain matrix structure. To deal with this, one makes repeated use of a Bethe ansatz. For this reason this approach is referred to as the nested Bethe ansatz.

In this chapter we will apply the nested Bethe ansatz to bound states of the $\text{AdS}_5 \times \text{S}^5$ superstring. However, the Hamiltonian is not explicitly known. Nevertheless, one can apply the Bethe ansatz procedure by using the explicit S-matrix and the dispersion relation. We will first exemplify the nested Bethe ansatz procedure in the nonlinear Schrödinger model, highlighting some of its features, before moving to the string model.

7.1 Formalism

We will set up the machinery of the Bethe ansatz in the context of field theories. We start by discussing the nonlinear Schrödinger model which is a theory without internal degrees of freedom. After this we will add internal degrees of freedom and discuss the nesting procedure.

7.1.1 Nonlinear Schrödinger Model

Let us first discuss the Bethe ansatz in the nonlinear Schrödinger model. An excellent review on this subject is given in [148]. The nonlinear Schrödinger model is defined by the following Hamiltonian operator

$$H = \int dx [\partial_x \phi^* \partial_x \phi + c \phi^* \phi^* \phi \phi], \quad (7.1)$$

where ϕ is a bosonic field with canonical equal time commutation relations

$$[\phi(x), \phi^*(y)] = \delta(x - y). \quad (7.2)$$

This model is integrable and the Hamiltonian commutes with the number operator $N = \int dx \phi^* \phi$ (the number of particles is preserved). This means that we can consider the different N -body states of the Hilbert space separately. For simplicity we will restrict to two-particle wave functions.

The Bethe ansatz is a plane-wave type ansatz for the eigenstates of the Hamiltonian. Consider the system on a line. Suppose we have two particles with momenta k_1, k_2 at positions x_1, x_2 that are well-separated. If $x_1 < x_2$, the particles do not interact and the natural guess for the wave function of this configuration would be

$$|\Psi(k_1, k_2)\rangle_1 = \int dx_1 dx_2 \theta(x_1 < x_2) e^{i(k_1 x_1 + k_2 x_2)} \phi^*(x_1) \phi^*(x_2) |0\rangle, \quad (7.3)$$

where θ is the heaviside step function

$$\theta(x < y) \equiv \theta(y - x) = \begin{cases} 1 & \text{if } x < y, \\ \frac{1}{2} & \text{if } x = y, \\ 0 & \text{else.} \end{cases} \quad (7.4)$$

Similarly for $x_1 > x_2$ one would write

$$|\Psi(k_1, k_2)\rangle_2 = \int dx_1 dx_2 \theta(x_2 < x_1) e^{i(k_1 x_1 + k_2 x_2)} \phi^*(x_1) \phi^*(x_2) |0\rangle. \quad (7.5)$$

Of course when traversing from $x_1 < x_2$ to $x_1 > x_2$ the particles come within interaction range and should interact according to the S-matrix. This leads us to the following ansatz for the total eigenstate

$$\begin{aligned} |\Psi(k_1, k_2)\rangle &= |\Psi(k_1, k_2)\rangle_1 + \mathcal{A} |\Psi(k_1, k_2)\rangle_2 \\ &= \int dx_1 dx_2 [\theta(x_1 < x_2) + \theta(x_2 < x_1) \mathcal{A}] e^{i(k_1 x_1 + k_2 x_2)} \phi^*(x_1) \phi^*(x_2) |0\rangle, \end{aligned} \quad (7.6)$$

where the coefficient \mathcal{A} should capture the scattering information.

Since $|\Psi(k_1, k_2)\rangle$ should be an eigenstate of the Hamiltonian, we will find restrictions on \mathcal{A} . By the canonical commutation rules (7.2) one finds

$$[H, \phi^*] = -\partial^2 \phi^* + 2c\phi^* \phi^* \phi. \quad (7.7)$$

From this it is easily deduced that

$$\begin{aligned} H|\Psi(k_1, k_2)\rangle &= \int dx_1 dx_2 [\theta(x_2 - x_1) + \theta(x_1 - x_2)\mathcal{A}] e^{i(k_1 x_1 + k_2 x_2)} \times \\ &\times (2c\phi^*(x_1)\phi^*(x_2)\delta(x_1 - x_2) - \partial^2 \phi^*(x_1)\phi^*(x_2) - \phi^*(x_1)\partial^2 \phi^*(x_2)) |0\rangle. \end{aligned} \quad (7.8)$$

Let us first work out the term proportional to c . By using the relation $\delta(x)\theta(x) = \frac{1}{2}\delta(x)$ this term reduces to

$$(1 + \mathcal{A})c \int dx e^{i(k_1 + k_2)x} \phi^*(x)\phi^*(x)|0\rangle. \quad (7.9)$$

The derivative term in the above expression can be evaluated by repeated partial integrating. Let us focus on the term $\partial^2 \phi^*(x_1)\phi^*(x_2)$

$$\begin{aligned} &\int dx_1 dx_2 [\theta(x_2 - x_1) + \theta(x_1 - x_2)\mathcal{A}] e^{i(k_1 x_1 + k_2 x_2)} \partial^2 \phi^*(x_1)\phi^*(x_2)|0\rangle \\ &= - \int dx_1 dx_2 i k_1 [\theta(x_2 - x_1) + \theta(x_1 - x_2)\mathcal{A}] e^{i(k_1 x_1 + k_2 x_2)} \partial \phi^*(x_1)\phi^*(x_2)|0\rangle + \\ &\quad - \int dx_1 dx_2 (\mathcal{A} - 1) \delta(x_1 - x_2) e^{i(k_1 x_1 + k_2 x_2)} \partial \phi^*(x_1)\phi^*(x_2)|0\rangle \\ &= \int dx_1 dx_2 -k_1^2 [\theta(x_2 - x_1) + \theta(x_1 - x_2)\mathcal{A}] e^{i(k_1 x_1 + k_2 x_2)} \phi^*(x_1)\phi^*(x_2)|0\rangle + \\ &\quad + \int dx_1 dx_2 i k_1 (\mathcal{A} - 1) \delta(x_2 - x_1) e^{i(k_1 x_1 + k_2 x_2)} \phi^*(x_1)\phi^*(x_2)|0\rangle + \\ &\quad - \int dx (\mathcal{A} - 1) e^{i(k_1 + k_2)x} \frac{1}{2} \partial (\phi^*(x))^2 |0\rangle \\ &= -k_1^2 \int dx_1 dx_2 [\theta(x_2 - x_1) + \theta(x_1 - x_2)\mathcal{A}] e^{i(k_1 x_1 + k_2 x_2)} \phi^*(x_1)\phi^*(x_2)|0\rangle + \\ &\quad + \int dx i(k_1 + \frac{k_1 + k_2}{2})(\mathcal{A} - 1) e^{i(k_1 + k_2)x} \phi^*(x)\phi^*(x)|0\rangle \end{aligned} \quad (7.10)$$

where we used $\partial_x \theta(x) = \delta(x)$ and applied partial integration in all the steps. The other derivative term in (7.8) can be computed analogously and we obtain

$$\begin{aligned} H|\Psi(k_1, k_2)\rangle &= (k_1^2 + k_2^2)|\Psi(k_1, k_2)\rangle + \\ &+ \{c(1 + \mathcal{A}) + i(k_1 - k_2)(1 - \mathcal{A})\} \int dx e^{i(k_1 + k_2)x} \phi^*(x)\phi^*(x)|0\rangle. \end{aligned} \quad (7.11)$$

From this one finds that $|\Psi(k_1, k_2)\rangle$ is an eigenstate of the Hamiltonian provided that

$$i(k_1 - k_2)(1 - \mathcal{A}) + c(1 + \mathcal{A}) = 0, \quad (7.12)$$

which is solved by

$$\mathcal{A}(k_2, k_1) = \frac{k_2 - k_1 - ic}{k_2 - k_1 + ic}. \quad (7.13)$$

Concluding, with this choice for \mathcal{A} we have that $|\Psi(k_1, k_2)\rangle$ is an eigenstate of the Hamiltonian with eigenvalue

$$H|\Psi(k_1, k_2)\rangle = (k_1^2 + k_2^2)|\Psi(k_1, k_2)\rangle. \quad (7.14)$$

The value for \mathcal{A} , (7.13) is actually the two-particle S-matrix. This can be seen in a very intuitive way. Consider our ansatz with $k_1 > k_2$, then $|\Psi(k_1, k_2)\rangle_1$ has the interpretation of two particles that are going to collide. This means that for $k_1 > k_2$ one would consider $|\Psi(k_1, k_2)\rangle_1$ to be an in-state. Equivalently, for $k_2 > k_1$ this would be an out-state, from which the relation of \mathcal{A} with the S-matrix becomes apparent. It is now straightforward to extend this ansatz to more than two particles. Notice also that in case of vanishing interaction ($c = 0$) one finds $\mathcal{A} = 1$ and the eigenstate reduces to a genuine sum of plane-waves.

The above discussion is valid on an infinite line. There is no restriction on the values of the momenta k_1, k_2 and the spectrum is continuous. The next step is to consider periodic boundary conditions. Consider the system on a line of large length $L \rightarrow \infty$, then the wave function

$$\psi(x_1, x_2) := \langle 0 | \phi(x_1) \phi(x_2) | \Psi(k_1, k_2) \rangle \quad (7.15)$$

needs to be periodic

$$\psi(0, x_2) = \psi(L, x_2), \quad \psi(x_1, 0) = \psi(x_1, L). \quad (7.16)$$

One immediately sees that this reduces to the following equations

$$e^{ik_j L} = \prod_{i \neq j} \mathcal{A}(k_j, k_i), \quad (7.17)$$

which are the Bethe equations. They capture the $\frac{1}{L}$ corrections to the momenta. This can be seen by considering for example the case $c = 0$. Solving k will then give $k = 2n\pi/L$ for some integer n . One now obtains the eigenvalues of the Hamiltonian by solving (7.17) and plugging the solutions in equation (7.14).

We work in the regime $L \rightarrow \infty$ because we have freely made use of partial integration discarding all boundary terms. To this end we tacitly assumed that we worked with rapidly decreasing fields.

In short, in the coordinate Bethe ansatz one builds up an eigenstate of the Hamiltonian of the system by considering regions where the particles are separated. In each of these regions one makes an ansatz for the wave function consisting of a sum of plane waves. The coefficients

relating the different regions are described by the S-matrix of the system. This is exemplified in figure 7.1. The reasoning behind this ansatz is that if the particles are well-separated they do not feel interactions and behave as plane-waves [104]. When particles scatter they will cross regions and pick up the factor from the S-matrix. Since the system is integrable, only two-particle interactions will play a role. Periodicity is then imposed on these coefficients, resulting in a set of Bethe equations for the particle momenta k_i .

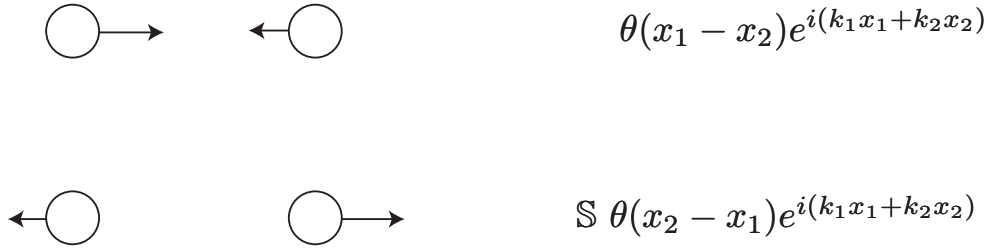


Figure 7.1: Schematic overview of the Bethe ansatz for two-particles. There are two regions for which one makes a plane-wave ansatz. The regions are denoted by the theta-functions and related via the S-matrix.

7.1.2 Adding color

We now consider a system with internal degrees of freedom, i.e. particles have color. As an example we will consider the system described by the Hamiltonian [148]

$$H = \int dx \sum_{a=1,2} \partial_x \phi_a^*(x) \partial_x \phi_a(x) + \sum_{a,b=1,2} \phi_a^*(x) \phi_b^*(x) \phi_a(x) \phi_b(x). \quad (7.18)$$

The natural way to generalize (7.6) would be to consider all possible types of orderings of momenta and positions of the particles. For two particles this becomes

$$|\Psi(k_1, k_2)\rangle = \int dx_1 dx_2 \psi(x_1, x_2) \phi_{a_1}^*(x_1) \phi_{a_2}^*(x_1) |0\rangle, \quad (7.19)$$

with

$$\psi(x_1, x_2) = \sum_{\mathcal{P}, \mathcal{Q}} \mathcal{A}^{\mathcal{P}|\mathcal{Q}} \theta(x_{\mathcal{Q}_1} < x_{\mathcal{Q}_2}) e^{i(k_{\mathcal{P}_1} x_{\mathcal{Q}_1} + k_{\mathcal{P}_2} x_{\mathcal{Q}_2})}, \quad (7.20)$$

where $\mathcal{A}^{\mathcal{P}|\mathcal{Q}}$ are constants, and \mathcal{P}, \mathcal{Q} are permutations of $\{1, 2\}$. Explicitly, we get

$$\begin{aligned} \psi(x_1, x_2) = & \theta(x_1 < x_2) \left\{ \mathcal{A}^{12|12} e^{i(k_1 x_1 + k_2 x_2)} + \mathcal{A}^{12|21} e^{i(k_2 x_1 + k_1 x_2)} \right\} + \\ & + \theta(x_2 < x_1) \left\{ \mathcal{A}^{21|12} e^{i(k_1 x_2 + k_2 x_1)} + \mathcal{A}^{21|21} e^{i(k_1 x_1 + k_2 x_2)} \right\} \end{aligned} \quad (7.21)$$

This wave function is continuous if we require that

$$\mathcal{A}^{12|12} + \mathcal{A}^{12|21} = \mathcal{A}^{21|12} + \mathcal{A}^{21|21}. \quad (7.22)$$

By performing a similar computation as in the previous section, one finds that $|\Psi(k_1, k_2)\rangle$ is a continuous eigenstate of the Hamiltonian provided the coefficients satisfy

$$\begin{pmatrix} \mathcal{A}^{12|21} \\ \mathcal{A}^{21|21} \end{pmatrix} = -\frac{\lambda_{12}}{1 + \lambda_{12}} \begin{pmatrix} \mathcal{A}^{12|12} \\ \mathcal{A}^{21|12} \end{pmatrix} + \frac{1}{1 + \lambda_{12}} \begin{pmatrix} \mathcal{A}^{21|12} \\ \mathcal{A}^{12|12} \end{pmatrix}, \quad \lambda_{ij} = \frac{ic}{k_i - k_j}. \quad (7.23)$$

The eigenvalue of $|\Psi(k_1, k_2)\rangle$ is

$$H|\Psi(k_1, k_2)\rangle = (k_1^2 + k_2^2)|\Psi(k_1, k_2)\rangle. \quad (7.24)$$

We would once again like to interpret the different terms in the wave function as in- and out-states. To this end it is convenient to make the change of variables $x_1 \leftrightarrow x_2$ in the terms proportional to $\mathcal{A}^{12|21}, \mathcal{A}^{21|12}$ to obtain

$$\begin{aligned} |\Psi(k_1, k_2)\rangle &= \int dx_1 dx_2 \\ &e^{i(k_1 x_1 + k_2 x_2)} \theta(x_1 < x_2) \left\{ \mathcal{A}^{12|12} \phi_{a_1}^*(x_1) \phi_{a_2}^*(x_2) + \mathcal{A}^{21|12} \phi_{a_2}^*(x_1) \phi_{a_1}^*(x_2) \right\} |0\rangle + \\ &+ e^{i(k_1 x_1 + k_2 x_2)} \theta(x_2 < x_1) \left\{ \mathcal{A}^{21|21} \phi_{a_1}^*(x_1) \phi_{a_2}^*(x_2) + \mathcal{A}^{12|21} \phi_{a_2}^*(x_1) \phi_{a_1}^*(x_2) \right\} |0\rangle. \end{aligned} \quad (7.25)$$

In this expression one recognizes the first term as an in-state, which is a superposition of a wave where the first particle has color a_1 and the second a_2 and of a wave with the color indices interchanged. The different coefficients from (7.23) then admit an interpretation as being coefficients of the S-matrix. The permutation \mathcal{Q} in coefficients $\mathcal{A}^{\mathcal{P}|\mathcal{Q}}$ actually labels the color distribution. To make this concrete let us explicitly include these indices, i.e. we write

$$\begin{aligned} \mathcal{A}_{a_1 a_2}^{12} &\equiv \mathcal{A}^{12|12}, & \mathcal{A}_{a_2 a_1}^{12} &\equiv \mathcal{A}^{21|12}, \\ \mathcal{A}_{a_1 a_2}^{21} &\equiv \mathcal{A}^{21|21}, & \mathcal{A}_{a_2 a_1}^{21} &\equiv \mathcal{A}^{12|21}. \end{aligned} \quad (7.26)$$

There are two distinct cases to be considered, namely $a_1 = a_2$ and $a_1 \neq a_2$. For the case $a_1 = a_2$ it is readily seen that both $\mathcal{A}^{12|12}$ and $\mathcal{A}^{21|12}$ describe the same state, which is made particularly clear by our notation above (7.26). Hence in this case our state is described by just two components $\mathcal{A}^{12|12} = \mathcal{A}_{a_1 a_1}^{12}$ and $\mathcal{A}^{21|21} = \mathcal{A}_{a_1 a_1}^{21}$. One finds that (7.23) implies $\mathcal{A}^{21|21} = \frac{1-\lambda_{12}}{1+\lambda_{12}} \mathcal{A}^{12|12}$, which agrees with the result (7.13) from the previous section. Define the following S-matrix on $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$\mathbb{S} = \frac{\mathbb{1} - \lambda_{12} P}{1 + \lambda_{12}}, \quad \mathbb{S} = \mathbb{S}_{kl}^{ij} E_k^i \otimes E_l^j, \quad (7.27)$$

where P is the permutation operator and E_j^i are the standard matrix unities, that have all zeroes except for a 1 at position (j, i) . The S-matrix \mathbb{S} can easily be shown to satisfy unitarity and the Yang-Baxter equation. From (7.23) we find that $|\Psi(k_1, k_2)\rangle$ is an eigenfunction of the Hamiltonian provided that

$$\mathcal{A}_{a_1 a_2}^{21} = \mathbb{S} \cdot \mathcal{A}_{a_1 a_2}^{12} = \mathbb{S}_{a_1 a_2}^{b_1 b_2} \mathcal{A}_{b_1 b_2}^{12}, \quad (7.28)$$

where repeated indices are summed over. Because of this, specifying the vector $\begin{pmatrix} \mathcal{A}_{a_1 a_2}^{12} \\ \mathcal{A}_{a_2 a_1}^{12} \end{pmatrix}$ fixes the wave function completely.

The generalization to N particles is now obvious; one takes

$$\psi = \sum_{\mathcal{P}, \mathcal{Q}} \mathcal{A}^{\mathcal{P}|\mathcal{Q}} \theta(x_{\mathcal{Q}_1} < \dots < x_{\mathcal{Q}_N}) e^{i \sum_i k_{\mathcal{P}_i} x_{\mathcal{Q}_i}}. \quad (7.29)$$

The coefficients of the vectors $\xi_{\mathcal{P}} \equiv (\mathcal{A}_{a_1 \dots a_N}^{\mathcal{P}})$ are then related by (7.28)

$$\mathcal{A}_{a_1 \dots a_N}^{\mathcal{P}} = \mathbb{S}_{ii+1} \cdot \mathcal{A}_{a_1 \dots a_N}^{(ii+1)\mathcal{P}}, \quad (7.30)$$

where $(ii+1)$ is the 2-cycle that permutes i and $i+1$. Hence the state is fixed by specifying the vector $\xi_0 \equiv (\mathcal{A}_{a_1 \dots a_N}^{\mathbb{1}})$, where $\mathbb{1}$ stands for the trivial permutation. The periodic boundary conditions now become the following matrix equation

$$e^{ik_j L} \xi_0 = \vec{\mathbb{S}}_j \xi_0 \equiv \mathbb{S}_{j+1,j} \mathbb{S}_{j+2,j} \dots \mathbb{S}_{N,j} \mathbb{S}_{1,j} \mathbb{S}_{2,j} \dots \mathbb{S}_{j-1,j} \xi_0. \quad (7.31)$$

From the Yang-Baxter equation it is readily checked that $[\vec{\mathbb{S}}_i, \vec{\mathbb{S}}_j] = 0$. This means that we can simultaneously diagonalize these matrix operators. To cope with the matrix structure in (7.31) one introduces a so-called nested structure. We will do this by finding the eigenvectors $\xi(N, M)$ of $\vec{\mathbb{S}}_j$ and then require that their eigenvalues $\Lambda(N, M)$ satisfy $\Lambda(N, M) = e^{ik_j L}$.

In our system we consider particles of two different species for simplicity, created by ϕ_1^*, ϕ_2^* respectively (the indices label the color). It is readily seen that the S-matrix preserves the number of particles of type 2 (and hence also of type 1). This means that there are two numbers that determine an eigenspace of $\vec{\mathbb{S}}_j$, namely the total number of particles N and the number of particles of type 2, denoted by M . This reminds of the XXX spin chain where M would be the number of spin up particles in a background of spin down particles. In analogy to this, we will identify ξ_0 with states of the XXX spin chain and treat M as excitations thereon.

Consider a spin chain with N sites $\bigotimes_{i=1}^N \mathbb{C}^2$. To any distribution of particles, i.e. any component of ξ_0 , we can associate a state of this chain and vice versa

$$\mathcal{A}_{a_1 \dots a_N}^{\mathbb{1}} = \mathcal{A}_{a_1 \dots a_N}^{\mathbb{1}} e_{a_1} \otimes \dots \otimes e_{a_N}, \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (7.32)$$

here repeated indices are not summed over. By (7.28) we see that the action of the S-matrix on ξ_0 is exactly the same as the action of the S-matrix on the spin chain. Thus, the diagonalization of $\vec{\mathbb{S}}_j$ is reduced to a problem on the spin chain. Therefore we will solve (7.31) on the chain introduced above. Let us stress that the introduced N -site spin chain is more of a practical bookkeeping device to deal with the index structure than a real physical system.

We will first deal with the cases $M = 0, 1, 2$ before presenting the general ansatz. For $M = 0$, all the particles are of the same color and we are back in the situation discussed in

previous section. There is only one independent component $\mathcal{A}_{1\dots 1}^{\mathbb{I}}$ and the rest are related to that via the scattering amplitude \mathbb{S}_{11}^{11} , e.g. $\mathcal{A}_{1\dots 1}^{(21)} = S_0(k_1, k_2)\mathcal{A}_{1\dots 1}^{\mathbb{I}}$, where we defined

$$S_0(k_i, k_j) = \mathbb{S}_{11}^{11}(k_i, k_j) = \frac{1 - \lambda_{ij}}{1 + \lambda_{ij}}. \quad (7.33)$$

Then (7.31) becomes

$$e^{ik_j L} = \prod_{i \neq j}^N S_0(k_i, k_j), \quad (7.34)$$

which indeed agrees with (7.17).

For $M = 1$ we make the following ansatz

$$\xi(N, 1) = \sum_i \Phi_i(y) \underbrace{e_1 \otimes \dots \otimes e_1}_{i-1} \otimes e_2 \otimes \underbrace{e_1 \dots \otimes e_1}_{N-i}, \quad \Phi_i(y) = f(y, k_i) \prod_{n=1}^{i-1} S(y, k_n), \quad (7.35)$$

where f, S are functions that are to be determined. The vector $\xi(N, 1)$ is then related to the actual ansatz (7.29) via equation (7.32). The function $\Phi_i(y)$ in the above ansatz describes an excitation with momentum y being moved to the place i on the chain. With a modest amount of foresight we have put in a parameter y explicitly as it will appear in the solutions of f, S later. The interpretation is that when it is moved there, it scatters with the different particles along the way, producing $i - 1$ scattering terms $S(y, k)$ and then it is inserted at position i that is described by the term $f(y, k_i)$.

Returning back to the two-particle case, let us see what restrictions one can put on the functions f, S . By relabelling $x_1 \leftrightarrow x_2$ in (7.25) one can map the in-state to the out-state. From this one sees it is natural to take

$$\mathcal{A}^{21|21}(k_1, k_2) \sim \mathcal{A}^{21|12}(k_2, k_1), \quad \mathcal{A}^{12|21}(k_1, k_2) \sim \mathcal{A}^{12|12}(k_2, k_1). \quad (7.36)$$

On the other hand, when $a_1 = a_2$ we know that the coefficients are related via $S_0(k_1, k_2)$. From this we find that one should require

$$\mathcal{A}^{21|21}(k_1, k_2) = S_0 \mathcal{A}^{21|12}(k_2, k_1), \quad \mathcal{A}^{12|21}(k_1, k_2) = S_0 \mathcal{A}^{12|12}(k_2, k_1). \quad (7.37)$$

In general for N sites, this is formulated as

$$\mathcal{A}_{a_1 \dots a_N}^{(ii+1)}(k_1, \dots, k_N) = S_0(k_i, k_{i+1}) \mathcal{A}_{a_1 \dots a_{i+1} a_i \dots a_N}^{\mathbb{I}}(k_1, \dots, k_{i+1}, k_i, \dots, k_N). \quad (7.38)$$

Via (7.23) one can then relate this to the S-matrix.

$$\mathbb{S}_{ii+1} \mathcal{A}_{a_1 \dots a_N}^{\mathbb{I}}(k_1, \dots, k_N) = S_0(k_i, k_{i+1}) \mathcal{A}_{a_1 \dots a_{i+1} a_i \dots a_N}^{\mathbb{I}}(k_1, \dots, k_{i+1}, k_i, \dots, k_N). \quad (7.39)$$

We see that vectors satisfying the above condition diagonalize \vec{S}_j ‘up to a permutation’. Restricting (7.35) to two particles ($N = 2$) then (7.39) amounts to the following equations

$$-\frac{\lambda_{12}}{1+\lambda_{12}}f(k_1, y) + \frac{1}{1+\lambda_{12}}f(k_2, y)S(k_1, y) = S_0(k_1, k_2)f(k_2, y) \quad (7.40)$$

$$\frac{1}{1+\lambda_{12}}f(k_1, y) - \frac{\lambda_{12}}{1+\lambda_{12}}f(k_2, y)S(k_1, y) = S_0(k_1, k_2)f(k_1, y)S(k_2, y). \quad (7.41)$$

These equations are uniquely solved by

$$f(k, y) = \frac{y}{k - y + \frac{ic}{2}}, \quad S(k, y) = \frac{k - y - \frac{ic}{2}}{k - y + \frac{ic}{2}}, \quad (7.42)$$

where y arises as an integration constant. Returning back to our original matrix equation (7.31) one can check that $\xi(N, 1)$ is an eigenstate provided

$$1 = \prod_{i=1}^N \frac{k_i - y_\alpha - \frac{ic}{2}}{k_i - y_\alpha + \frac{ic}{2}}. \quad (7.43)$$

The eigenvalue equation becomes

$$e^{ik_j L} = \frac{k_i - y - \frac{ic}{2}}{k_i - y + \frac{ic}{2}} \prod_{i \neq j} S_0(k_i, k_j). \quad (7.44)$$

Finally let us treat the case $M = 2$. In this case we introduce an auxiliary S-matrix \mathbb{S}^{II} that deals with the scattering of two auxiliary excitations on the spin chain with momenta y_1, y_2 . Explicitly for two excitations one makes the following ansatz

$$\begin{aligned} \xi(N, 2) = & \sum_{j < i} \Phi_i(y_1) \Phi_j(y_2) \underbrace{e_1 \otimes \dots \otimes e_1}_{j-1} \otimes e_2 \otimes \underbrace{e_1 \dots \otimes e_1}_{i-j-1} \otimes e_2 \otimes \underbrace{e_1 \dots \otimes e_1}_{N-i} + \\ & \mathbb{S}^{\text{II}} \sum_{j < i} \Phi_i(y_2) \Phi_j(y_1) \underbrace{e_1 \otimes \dots \otimes e_1}_{j-1} \otimes e_2 \otimes \underbrace{e_1 \dots \otimes e_1}_{i-j-1} \otimes e_2 \otimes \underbrace{e_1 \dots \otimes e_1}_{N-i}. \end{aligned} \quad (7.45)$$

We again require (7.39). By considering the restricting of the expression for $\xi(N, 2)$ to two sites one finds a unique solution \mathbb{S}^{II} of (7.39)

$$\mathbb{S}^{\text{II}}(y_1, y_2) = \frac{y_1 - y_2 - ic}{y_1 - y_2 + ic}. \quad (7.46)$$

In general we make the following ansatz for $\xi(N, M)$

$$\begin{aligned} \xi(N, M) = & \sum_{i_1 < \dots < i_M} \Phi_{i_1}(y_1) \dots \Phi_{i_M}(y_M) \underbrace{e_1 \otimes \dots \otimes e_1}_{i_1-1} \otimes e_2 \otimes e_1 \dots \otimes \dots + \\ & + \mathbb{S}_{12}^{\text{II}} \sum_{i_1 < \dots < i_M} \Phi_{i_1}(y_2) \Phi_{i_2}(y_1) \dots \Phi_{i_M}(y_M) \underbrace{e_1 \otimes \dots \otimes e_1}_{i_1-1} \otimes e_2 \otimes e_1 \otimes \dots + \\ & + \dots \end{aligned} \quad (7.47)$$

where ... stands for all terms with the auxiliary rapidities y_i permuted and multiplied with the appropriate factor. By using the explicit expression for $\xi(N, M)$, one can see that it is an eigenstate of \vec{S}_i from (7.31) if

$$1 = \prod_{i=1}^N \frac{k_i - y_\alpha - \frac{ic}{2}}{k_i - y_\alpha + \frac{ic}{2}} \prod_{\beta=1}^M \frac{y_\alpha - y_\beta - ic}{y_\alpha - y_\beta + ic}. \quad (7.48)$$

The complete solution of the eigenvalue equation (7.31) is now encoded in a set of coupled algebraic equations

$$e^{ik_j L} = \prod_{i \neq j}^N S_0(k_i, k_j) \prod_{\alpha=1}^M \frac{k_j - y_\alpha - \frac{ic}{2}}{k_j - y_\alpha + \frac{ic}{2}} \quad (7.49)$$

$$1 = \prod_{i=1}^N \frac{k_i - y_\alpha - \frac{ic}{2}}{k_i - y_\alpha + \frac{ic}{2}} \prod_{\beta=1}^M \frac{y_\alpha - y_\beta - ic}{y_\alpha - y_\beta + ic}. \quad (7.50)$$

Concluding we find that $|\Psi(k_1, k_2)\rangle$ is an eigenfunction of the Hamiltonian H on a circle of circumference L with eigenvalue $\sum_{i=1}^N k_i^2$ provided the momenta k_i satisfy (7.49), (7.50).

7.2 The $\mathfrak{su}(2|2)$ (nested) coordinate Bethe Ansatz

The Hamiltonian for the $\text{AdS}_5 \times S^5$ superstring is not explicitly known. Therefore, one can not explicitly propose a state (7.25) and compute the action of the Hamiltonian on this. Even so, in the nonlinear Schrödinger model, integrability implies that the coefficients of the eigenstate are related via the S-matrix and the eigenvalue is described by the dispersion relation $H(k) = k^2$.

Luckily, for the $\text{AdS}_5 \times S^5$ superstring both the S-matrix and the dispersion relation are explicitly known from symmetries. Thus we will use these objects to construct our Bethe ansatz rather than the explicit Hamiltonian. The dispersion relation is given by [46]

$$H(p) = \sqrt{1 + 4g^2 \sin^2 \frac{p}{2}} \quad (7.51)$$

and the S-matrix was explicitly derived in chapter 4. The full S-matrix contains two copies of the \mathfrak{h} invariant S-matrix, cf. section 4.9. We will first restrict to just one copy of the invariant S-matrix.

Mimicking the discussion from the previous section, we divide the space into asymptotic regions $\mathcal{P}|\mathcal{Q}$. In these regions we make the following ansatz

$$\psi = \sum_{\mathcal{P}, \mathcal{Q}} \mathcal{A}^{\mathcal{P}|\mathcal{Q}} \theta(x_{\mathcal{Q}_1} < \dots < x_{\mathcal{Q}_{K-1}}) e^{i \sum_i p_{\mathcal{P}_i} x_{\mathcal{Q}_i}}. \quad (7.52)$$

The wave functions in the different asymptotic regions are related via the S-matrix. Since we are dealing with closed strings, we impose periodicity, from which the BAE can be read off:

$$e^{ik_j L} \xi_0 = \vec{S}_j \xi_0 \equiv \mathbb{S}_{j+1,j} \mathbb{S}_{j+2,j} \dots \mathbb{S}_{K^1,j} \mathbb{S}_{1,j} \mathbb{S}_{2,j} \dots \mathbb{S}_{j-1,j} \xi_0, \quad (7.53)$$

where $\xi_0 = (\mathcal{A}_{a_1 \dots a_{K^I}}^{\mathbb{I}})$ is again the vector describing the in-state and this time the indices a_i run in the $4\ell_i$ dimensional bound state representation.

The auxiliary spin chain whose states can be identified with ξ_0 is obtained under the identification

$$\mathcal{A}_{a_1 \dots a_{K^I}}^{\mathbb{I}} \sim e_{a_1} \otimes \dots \otimes e_{a_{K^I}} \in V_{\ell_1}(p_1) \otimes \dots \otimes V_{\ell_{K^I}}(p_{K^I}), \quad (7.54)$$

where $V_{\ell_i}(p_i)$ is the bound state representation with bound state number ℓ_i and momentum p_i .

For simplicity, we will first discuss the Bethe ansatz for fundamental particles. This discussion is straightforwardly generalized to bound states. In the second part of this chapter we will discuss a slightly different approach to apply the Bethe ansatz procedure; we will employ the Yangian symmetry of the S-matrix to obtain the BAE rather than the S-matrix itself.

7.2.1 Solving for the coefficients

In chapter 4 it was shown that three quantum numbers K^I, K^{II} and K^{III} are preserved in scattering processes. K^I corresponds to the number of particles, K^{II} is the total number of fermions and K^{III} labels the number of fermions of type 4. This means that the eigenspaces of $\vec{\mathbb{S}}_j$ are labelled by these numbers. We will now construct the eigenvectors $\xi(K^I, K^{II}, K^{III})$ of $\vec{\mathbb{S}}_j$ in a way analogous to the nonlinear Schrödinger model.

Let us first define a ‘vacuum’ on the spin chain:

$$\xi(K, 0, 0) = |0\rangle = w_1^{(1)} \dots w_1^{(K)}, \quad (7.55)$$

where $w_1^{(i)}$ is the bosonic variable associated to the space $V_{\ell_i}(p_i)$ in the superspace formalism. We find that

$$\vec{\mathbb{S}}_j |0\rangle = \prod_{i \neq j} S_0(k_i, k_j) |0\rangle. \quad (7.56)$$

In the light of (7.53), this means that the state (7.55) is indeed an eigenstate of $\vec{\mathbb{S}}_j$ and the corresponding BAE give

$$e^{ik_j L} = \prod_{i \neq j} S_0(k_i, k_j). \quad (7.57)$$

The next thing to consider is the case where we have a fermionic excitation in this vacuum (the case in which the other boson is inserted is treated later on). We can treat $\xi(K, 1, 0)$ and $\xi(K, 1, 1)$ simultaneously by making an ansatz of the following form (cf. equation (7.35)):

$$\xi(K, 1) = |\alpha\rangle := \sum_i \Psi_i(y) w_1^{(1)} \dots \theta_\alpha^{(i)} \dots w_1^{(K)}, \quad \Psi_k(y) = f(y, p_k) \prod_{l < k} S^{II, I}(y, p_l), \quad (7.58)$$

where $\alpha = 3, 4$. We will denote $S(y, p) \equiv S^{\text{II}, \text{I}}(y, p)$. We must check whether this construction is well-defined in the sense that it respects (7.39).

Because of the factorization property of the S-matrix, it again suffices to restrict to a two-particle state. By reducing equation (7.39) to this case, one derives from the explicit form of the S-matrix (cf. equation (4.1)) the following equations

$$\begin{aligned} \frac{e^{i\frac{p_1}{2}}}{e^{i\frac{p_2}{2}}} \frac{\eta(p_2)}{\eta(p_1)} \frac{x_1^+ - x_1^-}{x_1^+ - x_2^-} f(y, p_1) + e^{i\frac{p_1}{2}} \frac{x_1^- - x_2^-}{x_1^+ - x_2^-} f(y, p_2) S(y, p_1) &= f(y, p_2) \\ e^{-i\frac{p_2}{2}} \frac{x_1^+ - x_2^+}{x_1^+ - x_2^-} f(y, p_1) + \frac{\eta(p_1)}{\eta(p_2)} \frac{x_2^+ - x_2^-}{x_1^+ - x_2^-} f(y, p_2) S(y, p_1) &= f(y, p_1) S(y, p_2). \end{aligned} \quad (7.59)$$

These equations can be solved explicitly and the solution is given by:

$$f(y, p_k) = \eta(p_k) \sqrt{\frac{x_k^-}{x_k^+}} \frac{y}{y - x_k^-} \sqrt{\frac{g\ell_k}{2}}, \quad S(y, p_k) = \sqrt{\frac{x_k^-}{x_k^+}} \frac{y - x_k^+}{y - x_k^-}. \quad (7.60)$$

With a modest amount of foresight, we fix the overall normalization of f to be dependent on the bound state number ℓ_k .

The problem becomes more involved when inserting two excitations. In this case, one again introduces an auxiliary S-matrix, \mathbb{S}^{II} , that deals with interchanging excitations. The ansatz for the coefficient can be written as

$$|\alpha\beta\rangle = |\alpha\beta\rangle_{y_1 y_2} + \mathbb{S}^{\text{II}} \cdot |\alpha\beta\rangle_{y_1 y_2}, \quad (7.61)$$

where we define \mathbb{S}^{II} as

$$\mathbb{S}^{\text{II}} \cdot |\alpha\beta\rangle_{y_1 y_2} = M(y_1, y_2) |\alpha\beta\rangle_{y_2 y_1} + N(y_1, y_2) |\beta\alpha\rangle_{y_2 y_1}. \quad (7.62)$$

When $\alpha = \beta = 3$, we find

$$\begin{aligned} \xi(K, 2, 0) &= |33\rangle \\ &= \sum_{k < l} \Psi_k(y_1) \Psi_l(y_2) w_1^{(1)} \dots \theta_3^{(k)} \dots \theta_3^{(l)} \dots w_1^{(K)} + \\ &\quad + (M(y_1, y_2) + N(y_1, y_2)) \sum_{k < l} \Psi_k(y_2) \Psi_l(y_1) w_1^{(1)} \dots \theta_3^{(k)} \dots \theta_3^{(l)} \dots w_1^{(K)}, \end{aligned} \quad (7.63)$$

which is compatible with (7.39) if $M(y_1, y_2) + N(y_1, y_2) = -1$. In general we write

$$\begin{aligned} |\alpha\beta\rangle &= \sum_{k < l} \Psi_k(y_1) \Psi_l(y_2) w_1^{(1)} \dots \theta_\alpha^{(k)} \dots \theta_\beta^{(l)} \dots w_1^{(K)} + \\ &\quad + \mathbb{S}^{\text{II}} \cdot \sum_{k < l} \Psi_k(y_1) \Psi_l(y_2) w_1^{(1)} \dots \theta_\alpha^{(k)} \dots \theta_\beta^{(l)} \dots w_1^{(K)} + \\ &\quad + \epsilon^{\alpha\beta} \sum_k \Psi_k(y_1) \Psi_k(y_2) h(y_1, y_2, p_k) w_1^{(1)} \dots w_2^{(k)} \dots w_1^{(K)}. \end{aligned} \quad (7.64)$$

The term containing the w_2 variable needs to be included here since it has the same quantum numbers $K^{\text{II}}, K^{\text{III}}$ as the state $\theta_3\theta_4$ as was explained in chapter 4. We allow for a new function h , which describes the case when two fermions occupy the same site. This term also receives a contribution from \mathbb{S}^{II} , which for simplicity, we have absorbed in h .

Because of integrability we can restrict to just two sites. The above state $|\alpha\beta\rangle$ splits into the sum of a wave function with either zero, one or two fermions. The only new piece is the part containing two fermions which is given by

$$\begin{aligned} |\alpha\beta\rangle &= \{f(y_1, p_1)f(y_2, p_2)S(y_2, p_1) + Mf(y_2, p_1)f(y_1, p_2)S(y_1, p_1)\} \theta_\alpha^{(1)}\theta_\beta^{(2)} \\ &\quad + Nf(y_2, p_1)f(y_1, p_2)S(y_1, p_1)\theta_\beta^{(1)}\theta_\alpha^{(2)} \\ &\quad + \epsilon^{\alpha\beta}h(y_1, y_2, p_1)f(y_2, p_1)f(y_1, p_1)w_2^{(1)}w_1^{(2)} \\ &\quad + \epsilon^{\alpha\beta}h(y_1, y_2, p_2)f(y_2, p_2)f(y_1, p_2)S(y_2, p_1)S(y_1, p_1)w_1^{(1)}w_2^{(2)} \end{aligned} \quad (7.65)$$

Plugging this into (7.39) again allows one to find the explicit (unique) solutions of the unknown functions:

$$\begin{aligned} M(y_1, y_2) &= \frac{2i/g}{y_1 + \frac{1}{y_1} - y_2 - \frac{1}{y_2} - \frac{2i}{g}}, & N(y_1, y_2) &= -\frac{y_1 + \frac{1}{y_1} - y_2 - \frac{1}{y_2}}{y_1 + \frac{1}{y_1} - y_2 - \frac{1}{y_2} - \frac{2i}{g}} \\ h(y_1, y_2, p_k) &= \frac{i}{\ell_k \eta(p_k)^2} \frac{y_1 y_2 - x_k^+ x_k^-}{y_1 y_2} \frac{x_k^+ - x_k^-}{x_k^-} \frac{y_1 - y_2}{y_1 + \frac{1}{y_1} - y_2 - \frac{1}{y_2} - \frac{2i}{g}} \end{aligned} \quad (7.66)$$

Upon defining $v = y + \frac{1}{y}$ one finds

$$M = \frac{2i/g}{v_1 - v_2 - 2i/g}, \quad N = -\frac{v_1 - v_2}{v_1 - v_2 - 2i/g}. \quad (7.67)$$

From this we see that (7.28) and \mathbb{S}^{II} are basically the same operator. It is readily seen that in order for $|\alpha\beta\rangle$ to be an eigenvector of \vec{S}_j one has to require that

$$\mathbb{S}^{\text{II}}|\alpha\beta\rangle = |\beta\alpha\rangle, \quad (7.68)$$

which just indicates we should apply the procedure a second time to deal with the index structure associated with the fermionic indices $\alpha\beta$.

One can repeat the discussion for the nonlinear Schrödinger model to deal with this since (7.28) agrees with \mathbb{S}^{II} [38, 49]. In this process we are lead to introduce functions $S^{\text{II},\text{II}}, f^{(2)}, S^{\text{III},\text{II}}, S^{\text{III},\text{III}}$ similar to the f, S introduced for the nonlinear Schrödinger model in the previous section. The result of this consideration is

$$\begin{aligned} S^{\text{II},\text{II}} &= -M - N = 1, & f^{(2)}(w, y_k) &= \frac{w - \frac{i}{g}}{w - v_k - \frac{i}{g}} \\ S^{\text{III},\text{II}}(w, y_k) &= \frac{w - v_k + \frac{i}{g}}{w - v_k - \frac{i}{g}}, & S^{\text{III},\text{III}}(w_1, w_2) &= \frac{w_1 - w_2 - \frac{2i}{g}}{w_1 - w_2 + \frac{2i}{g}}. \end{aligned} \quad (7.69)$$

By putting all of this together one obtains the Bethe equations describing the large volume spectrum of the $\text{AdS}_5 \times \text{S}^5$ superstring [38, 47–49]:

$$\begin{aligned}
e^{ip_k L} &= \prod_{l=1, l \neq k}^{K^I} \left[S_0(p_k, p_l) \frac{x_k^+ - x_l^-}{x_k^- - x_l^+} \sqrt{\frac{x_l^+ x_k^-}{x_l^- x_k^+}} \right]^2 \prod_{\alpha=1}^2 \prod_{l=1}^{K_{(\alpha)}^{II}} \frac{x_k^- - y_l^{(\alpha)}}{x_k^+ - y_l^{(\alpha)}} \sqrt{\frac{x_k^+}{x_k^-}} \\
1 &= \prod_{l=1}^{K^I} \frac{y_k^{(\alpha)} - x_l^+}{y_k^{(\alpha)} - x_l^-} \sqrt{\frac{x_k^-}{x_k^+}} \prod_{l=1}^{K_{(\alpha)}^{II}} \frac{y_k^{(\alpha)} + \frac{1}{y_k^{(\alpha)}} - w_l^{(\alpha)} + \frac{i}{g}}{y_k^{(\alpha)} + \frac{1}{y_k^{(\alpha)}} - w_l^{(\alpha)} - \frac{i}{g}} \\
1 &= \prod_{l=1}^{K_{(\alpha)}^{II}} \frac{w_k^{(\alpha)} - y_k^{(\alpha)} - \frac{1}{y_k^{(\alpha)}} + \frac{i}{g}}{w_k^{(\alpha)} - y_k^{(\alpha)} - \frac{1}{y_k^{(\alpha)}} - \frac{i}{g}} \prod_{l \neq k}^{K_{(\alpha)}^{II}} \frac{w_k^{(\alpha)} - w_l^{(\alpha)} - \frac{2i}{g}}{w_k^{(\alpha)} - w_l^{(\alpha)} + \frac{2i}{g}},
\end{aligned} \tag{7.70}$$

where $\alpha = 1, 2$ reinstates the two independent copies of $\mathfrak{su}(2|2)$ and $S_0(p_k, p_l)$ is the overall phase of the S-matrix.

One can straightforwardly apply the same procedure to derive the spectrum for bound states by using the explicit bound state S-matrix. However, in the next section we choose a slightly different approach according to which the Bethe equations can be derived without reference to the explicit S-matrix.

7.3 Bethe Ansatz and Yangian Symmetry

In this section we will generalize the above construction to arbitrary bound states. We will do this by considering coproducts of (Yangian) symmetry generators. This formulation allows us to solve (7.39) without referring to the explicit form of the bound state S-matrix, but rather use its underlying symmetry.

7.3.1 Single excitations

We will again start by considering a single excitation on the vacuum

$$|0\rangle = (w_1^{(1)})^{\ell_1} \dots (w_1^{(K)})^{\ell_K}. \tag{7.71}$$

The natural generalization of a single excitation wave function (7.58) is:

$$\begin{aligned}
|\alpha\rangle &:= \sum_i \Psi_i(y) (w_1^{(1)})^{\ell_1} \dots \theta_\alpha^{(i)} (w_1^{(i)})^{\ell_i-1} \dots (w_1^{(K)})^{\ell_K}, \\
\Psi_k(y) &= f(y, p_k) \prod_{l < k} S^{I,II}(y, p_l),
\end{aligned} \tag{7.72}$$

As noted above, it suffices to restrict to two bound state representations, for which the wave function is of the form

$$|\alpha\rangle = f(p_1) \theta_\alpha^{(1)} (w_1^{(1)})^{\ell_1-1} (w_1^{(2)})^{\ell_2} + f(p_2) S(p_1) (w_1^{(1)})^{\ell_1} \theta_\alpha^{(2)} (w_1^{(2)})^{\ell_2-1}. \tag{7.73}$$

The remarkable fact is that one can write this as:

$$\tilde{\Delta}\mathbb{Q}_\alpha^1|0\rangle := \left(K_0(p_1, p_2)\Delta\mathbb{Q}_\alpha^1 + K_1(p_1, p_2)\Delta\hat{\mathbb{Q}}_\alpha^1\right)|0\rangle, \quad (7.74)$$

with

$$\begin{aligned} K_0 &= -\sqrt{\frac{2}{g}} \frac{x_2^- \{x_1^- x_2^- (x_1^+ x_2^+ - 1) - x_1^+ x_2^+ (1 + x_1^- [x_1^- + x_1^+ + x_2^+])\}}{(x_2^- - x_1^+)(2x_1^+ x_2^+ x_1^- x_2^- - x_1^- x_2^- - x_1^+ x_2^+)} \times \\ &\quad \times \left[\frac{f(p_2)S(p_1)}{\sqrt{\ell_2}\eta(p_2)} - \frac{e^{-i\frac{p_2}{2}} f(p_1)}{\sqrt{\ell_1}\eta(p_1)} \right] + \frac{e^{-i\frac{p_2}{2}} f(p_1)}{\sqrt{\ell_1}\eta(p_1)} \\ K_1 &= \frac{4i\sqrt{2}}{g^{3/2}} \frac{x_1^- x_2^- x_1^+ x_2^+}{(x_2^- - x_1^+)(2x_1^+ x_2^+ x_1^- x_2^- - x_1^- x_2^- - x_1^+ x_2^+)} \left[\frac{f(p_2)S(p_1)}{\sqrt{\ell_2}\eta(p_2)} - \frac{e^{-i\frac{p_2}{2}} f(p_1)}{\sqrt{\ell_1}\eta(p_1)} \right] \end{aligned} \quad (7.75)$$

For the moment let us keep f, S arbitrary. The invariance of the S-matrix under Yangian symmetry means that

$$\mathbb{S}\Delta\mathbb{Q}_\alpha^1 = \Delta^{op}\mathbb{Q}_\alpha^1\mathbb{S}, \quad \mathbb{S}\Delta\hat{\mathbb{Q}}_\alpha^1 = \Delta^{op}\hat{\mathbb{Q}}_\alpha^1\mathbb{S}. \quad (7.76)$$

In other words, we find:

$$\begin{aligned} \mathbb{S}|\alpha\rangle &= \mathbb{S}\left(K_0(p_1, p_2)\Delta\mathbb{Q}_\alpha^1 + K_1(p_1, p_2)\Delta\hat{\mathbb{Q}}_\alpha^1\right)|0\rangle \\ &= \left(K_0(p_1, p_2)\Delta^{op}\mathbb{Q}_\alpha^1 + K_1(p_1, p_2)\Delta^{op}\hat{\mathbb{Q}}_\alpha^1\right)\mathbb{S}|0\rangle \end{aligned} \quad (7.77)$$

$$= S_0\left(K_0(p_1, p_2)\Delta^{op}\mathbb{Q}_\alpha^1 + K_1(p_1, p_2)\Delta^{op}\hat{\mathbb{Q}}_\alpha^1\right)|0\rangle. \quad (7.78)$$

since $\mathbb{S}|0\rangle = S_0|0\rangle$. However, on the right hand side of (7.39) we find the coefficient with indices and momenta interchanged, which we denote by $|\alpha\rangle_\pi$. It is readily seen that

$$|\alpha\rangle_\pi = \left(K_0(p_2, p_1)\Delta^{op}\mathbb{Q}_\alpha^1 + K_1(p_2, p_1)\Delta^{op}\hat{\mathbb{Q}}_\alpha^1\right)|0\rangle. \quad (7.79)$$

This means that (7.39) corresponds to requiring that K_0 and K_1 are symmetric under interchanging $p_1 \leftrightarrow p_2$. In other words to find our coefficients, we have to solve

$$K_0(p_1, p_2) = K_0(p_2, p_1), \quad K_1(p_1, p_2) = K_1(p_2, p_1), \quad (7.80)$$

for the functions f and S .

From the explicit expressions for K_0, K_1 it is straightforward to prove that (7.80) is equivalent to the equations:

$$\begin{aligned} Kf(p_1) + Gf(p_2)S(p_1) &= f(p_2) \\ Lf(p_1) + Hf(p_2)S(p_1) &= f(p_1)S(p_2), \end{aligned} \quad (7.81)$$

with

$$\begin{aligned} K &= \frac{e^{i\frac{p_1}{2}} \sqrt{\ell_2 \eta(p_2)} x_1^+ - x_1^-}{e^{i\frac{p_2}{2}} \sqrt{\ell_1 \eta(p_1)} x_1^+ - x_2^-} & G &= e^{i\frac{p_1}{2}} \frac{x_1^- - x_2^-}{x_1^+ - x_2^-} \\ L &= e^{-i\frac{p_2}{2}} \frac{x_1^+ - x_2^+}{x_1^+ - x_2^-} & H &= \frac{\sqrt{\ell_1 \eta(p_1)} x_2^+ - x_2^-}{\sqrt{\ell_2 \eta(p_2)} x_1^+ - x_2^-}. \end{aligned} \quad (7.82)$$

These equations are solved by the f, S found before, i.e. we again find (7.60) as unique solution. Moreover, notice that from this construction we can read off the elements of the S-matrix. Namely, we rediscover in this way our coefficients describing the scattering of Case II states $\mathcal{Y}_k^{k,0}$, cf. section 4.8.

In conclusion, Yangian symmetry uniquely fixes the form of our wave function. We can now write the wave function, restricted to two sites, completely in terms of coproducts and, as a consequence, (7.39) is automatically satisfied. Finally, the explicit expressions for K_0, K_1 are

$$\begin{aligned} K_0(p_1, p_2, y) &= \sqrt{\frac{x_1^-}{x_1^+}} \sqrt{\frac{x_2^-}{x_2^+}} \frac{y}{(y - x_1^-)(y - x_2^-)} \left[y - \frac{x_1^- x_2^- x_1^+ x_2^+ (x_1^- + x_2^- + x_1^+ + x_2^+)}{2x_1^- x_2^- x_1^+ x_2^+ - x_1^- x_2^- - x_1^+ x_2^+} \right] \\ K_1(p_1, p_2, y) &= \frac{4i}{g} \sqrt{\frac{x_1^-}{x_1^+}} \sqrt{\frac{x_2^-}{x_2^+}} \frac{y}{(y - x_1^-)(y - x_2^-)} \left[\frac{x_1^- x_2^- x_1^+ x_2^+}{2x_1^- x_2^- x_1^+ x_2^+ - x_1^- x_2^- - x_1^+ x_2^+} \right]. \end{aligned} \quad (7.83)$$

This consideration is valid for *any* bound state numbers and hence wave function (7.72) is valid for *any* bound state representations. In particular, all bound state representations share the same function $S^{\text{I,II}}$.

7.3.2 Multiple excitations

When dealing with two excitations, one needs to introduce a level II S-matrix that deals with interchanging y_1 and y_2 .

Fundamental representations

Let us first restrict to fundamental representations and reformulate this in terms of coproducts. We will then move on to generic bound states.

The wave function was of the form

$$|\alpha\beta\rangle = |\alpha\beta\rangle_{y_1 y_2} + \mathbb{S}^{\text{II}} |\alpha\beta\rangle_{y_1 y_2}, \quad (7.84)$$

where

$$\mathbb{S}^{\text{II}} |\alpha\beta\rangle_{y_1 y_2} = M(y_1, y_2) |\alpha\beta\rangle_{y_2 y_1} + N(y_1, y_2) |\beta\alpha\rangle_{y_2 y_1}. \quad (7.85)$$

This state contained both fermions and bosons w_2 , so the natural way to write this would be:

$$|\alpha\beta\rangle_{y_1 y_2} = \left\{ (\tilde{\Delta}_{y_1} \mathbb{Q}_\alpha^1) (\tilde{\Delta}_{y_2} \mathbb{Q}_\beta^1) + \epsilon_{\alpha\beta} \Delta'_{y_1, y_2} \mathbb{L}_2^1 \right\} |0\rangle, \quad (7.86)$$

with

$$\Delta'_{y_1, y_2} \mathbb{L}_2^1 := L_0(y_1, y_2, p_1, p_2) \Delta \mathbb{L}_2^1 + L_1(y_1, y_2, p_1, p_2) \Delta \hat{\mathbb{L}}_2^1. \quad (7.87)$$

By taking $\alpha = \beta$, one easily checks that we reproduce the result from the previous chapter provided $M + N = -1$. Now we have to solve the coefficients L_0, L_1 such that our ansatz agrees with

$$\begin{aligned} & \{f(y_1, p_1)f(y_2, p_2)S(y_2, p_1) + Mf(y_2, p_1)f(y_1, p_2)S(y_1, p_1)\} \theta_\alpha^{(1)} \theta_\beta^{(2)} \\ & + Nf(y_2, p_1)f(y_1, p_2)S(y_1, p_1) \theta_\beta^{(1)} \theta_\alpha^{(2)} \\ & + \epsilon^{\alpha\beta} h(y_1, y_2, p_1) f(y_2, p_1) f(y_1, p_1) w_2^{(1)} w_1^{(2)} \\ & + \epsilon^{\alpha\beta} h(y_1, y_2, p_2) f(y_2, p_2) f(y_1, p_2) S(y_2, p_1) S(y_1, p_1) w_1^{(1)} w_2^{(2)}, \end{aligned} \quad (7.88)$$

where we keep the functions M, N, h arbitrary. The above expression consists of four independent terms which can be shown to give two equations for L_0 and two equations for L_1 . The next step is to impose that both L_0, L_1 are symmetric under the interchange $p_1 \leftrightarrow p_2$ in order to satisfy (7.39). This will give us four equations for the functions h, M, N which can be shown to be equivalent to the following set of equations:

$$\begin{aligned} \{f_{12}f_{21}S_{22} + Mf_{22}f_{11}S_{12}\} &= \{f_{11}f_{22}S_{21} + Mf_{21}f_{12}S_{11}\} \frac{D+E}{2} + Nf_{21}f_{12}S_{11} \frac{D-E}{2} \\ &\quad + (-f_{11}f_{21}h_{121} + f_{12}f_{22}S_{11}S_{21}h_{122}) \frac{C}{2} \\ Nf_{22}f_{11}S_{12} &= \{f_{11}f_{22}S_{21} + Mf_{21}f_{12}S_{11}\} \frac{D-E}{2} + Nf_{21}f_{12}S_{11} \frac{D+E}{2} \\ &\quad - (-f_{11}f_{21}h_{121} + f_{12}f_{22}S_{11}S_{21}h_{122}) \frac{C}{2}. \\ f_{11}f_{21}S_{12}S_{22}h_{121} &= \{f_{11}f_{22}S_{21} + (M-N)f_{21}f_{12}S_{11}\} \frac{F}{2} \\ &\quad + f_{11}f_{21}h_{121} \frac{1-B}{2} + f_{12}f_{22}S_{11}S_{21}h_{122} \frac{1+B}{2} \\ f_{12}f_{22}h_{122} &= -\{f_{11}f_{22}S_{21} + (M-N)f_{21}f_{12}S_{11}\} \frac{F}{2} \\ &\quad + f_{11}f_{21}h_{121} \frac{1+B}{2} + f_{12}f_{22}S_{11}S_{21}h_{122} \frac{1-B}{2}, \end{aligned} \quad (7.89)$$

where, for convenience, we introduced the short-hand notation $f_{kl} := f(y_k, p_l)$, $S_{kl} := S^{\Pi, I}(y_k, p_l)$, $M :=$

$M(y_1, y_2), N := N(y_1, y_2)$ and $h_{ijk} := h(y_i, y_j, p_k)$. The coefficients B, C, D, E, F are given by

$$\begin{aligned}
B &= \frac{2x_1^- x_2^- (x_2^+)^2 - (x_1^- x_2^- + 1)(x_2^- + x_1^+)x_2^+ + 2x_2^- x_1^+}{(1 - x_1^- x_2^-)(x_1^+ - x_2^-)x_2^+} \\
C &= 2i\eta(p_1)\eta(p_2) \frac{x_2^-}{x_2^+} \frac{e^{-\frac{ip_1}{2}}(x_2^+ - x_1^+)}{(1 - x_1^- x_2^-)(x_1^+ - x_2^-)} \\
D &= \frac{x_1^- - x_2^+}{x_2^- - x_1^+} \frac{e^{\frac{ip_1}{2}}}{e^{\frac{ip_2}{2}}} \\
E &= \frac{e^{\frac{ip_1}{2}}}{e^{\frac{ip_2}{2}}} \frac{(x_1^-(x_2^-(x_1^- - 2x_1^+) + 1)x_1^+ + (x_1^+ + x_1^-(x_2^- x_1^+ - 2))x_2^+)}{(1 - x_1^- x_2^-)(x_1^+ - x_2^-)x_1^+} \\
F &= 2i \frac{e^{-\frac{ip_1}{2}}}{\eta(p_1)\eta(p_2)} \frac{(x_1^+ - x_1^-)(x_2^+ - x_2^-)(x_2^+ - x_1^+)}{(1 - x_1^- x_2^-)(x_1^+ - x_2^-)}.
\end{aligned} \tag{7.90}$$

It is readily seen that these expressions coincide with elements from the fundamental S-matrix. Remarkably, these are exactly the same equations that one encounters in the nested Bethe Ansatz. In other words, the coefficients B, C, D, E, F indeed correspond to elements from the fundamental S-matrix and we again find (7.66) as the unique solution for M, N, h .

To conclude, let us give the explicit solutions for L_0, L_1 ,

$$\begin{aligned}
L_0 &= \frac{g(y_1 - y_2)x_1^- x_2^-}{2i(y_1 - x_1^-)(y_2 - x_1^-)(y_1 - x_2^-)(y_2 - x_2^-)} \times \\
&\times \left[(y_1 + y_2) - \frac{x_1^- x_2^- x_1^+ x_2^+ (x_1^- + x_2^- + x_1^+ + x_2^+)}{2x_1^+ x_2^+ x_1^- x_2^- - x_1^- x_2^- - x_1^+ x_2^+} - \frac{y_1 y_2 x_1^+ x_2^+}{2x_1^+ x_2^+ x_1^- x_2^- - x_1^- x_2^- - x_1^+ x_2^+} \left\{ \right. \right. \\
&\left. \left. (x_1^+ + x_2^+ - x_1^- - x_2^-)(x_1^- x_2^- - x_1^+ x_2^+) - \left(\frac{1}{x_2^-} + \frac{1}{x_1^+} + \frac{1}{x_2^+} + \frac{1}{x_1^-} \right) (x_1^- x_2^- + x_1^+ x_2^+) \right\} \right]
\end{aligned} \tag{7.91}$$

$$\begin{aligned}
L_1 &= \frac{y_1 y_2 x_1^- x_2^-}{(y_1 - x_1^-)(y_2 - x_1^-)(y_1 - x_2^-)(y_2 - x_2^-)} \times \\
&\times \left[(y_1 - y_2) + \frac{4ig^{-1}x_1^- x_2^- x_1^+ x_2^+}{2x_1^+ x_2^+ x_1^- x_2^- - x_1^- x_2^- - x_1^+ x_2^+} \right]
\end{aligned} \tag{7.92}$$

Note that they are indeed manifestly symmetric under $p_1 \leftrightarrow p_2$.

Bound states

When considering bound states there is a new term proportional to $\theta_3\theta_4$. To include this term we make the following generalization of the two excitation ansatz

$$\begin{aligned}
|\alpha\beta\rangle &= \sum_{k<l} \Psi_k(y_1)\Psi_l(y_2)(w_1^{(1)})^{\ell_1} \dots \theta_\alpha^{(k)}(w_1^{(k)})^{\ell_k-1} \dots \theta_\beta^{(l)}(w_1^{(l)})^{\ell_l-1} \dots (w_1^{(K)})^{\ell_K} + \\
&+ \mathbb{S}^{\text{II}} \cdot \sum_{k<l} \Psi_k(y_1)\Psi_l(y_2)(w_1^{(1)})^{\ell_1} \dots \theta_\alpha^{(k)}(w_1^{(k)})^{\ell_k-1} \dots \theta_\beta^{(l)}(w_1^{(l)})^{\ell_l-1} \dots (w_1^{(K)})^{\ell_K} + \\
&+ \epsilon^{\alpha\beta} \sum_k \Psi_k(y_1)\Psi_k(y_2)h(y_1, y_2, p_k)(w_1^{(1)})^{\ell_1} \dots w_2^{(k)}(w_1^{(k)})^{\ell_k-1} \dots (w_1^{(K)})^{\ell_K} \quad (7.93) \\
&+ \sum_k \Psi_k(y_1)\Psi_k(y_2)g(y_1, y_2, p_k)(w_1^{(1)})^{\ell_1} \dots \theta_\alpha^{(k)}\theta_\beta^{(k)}(w_2^{(1)})^{\ell_k-2} \dots (w_1^{(K)})^{\ell_K},
\end{aligned}$$

where the function g is again to be determined. Restricted to two sites we can try to match this with (7.86). This is indeed possible and imposing symmetry of L_0, L_1 as before provides equations that are uniquely solved by $g(y_1, y_2, p_k) = \frac{\ell_k-1}{2\ell_k}(1 + M(y_1, y_2) - N(y_1, y_2))$ and (7.66). Plugging these solutions back in L_0, L_1 , we find the same functions L_0, L_1 as in (7.91) but one has to bear in mind that x^\pm parameterize bound state solutions; they depend on the bound state number ℓ via equation(3.12).

By construction, this wave function satisfies (7.39) for any bound state S-matrix. Hence this solves our two excitation case. In particular one finds that also the level II S-matrix, \mathbb{S}^{II} is unchanged for bound states.

7.3.3 Bethe equations

By making use of coproducts and Yangian symmetry, we have found a way, independent of the explicit form of the S-matrix, to write down Bethe wave functions. This allowed us to find $S^{\text{II},\text{I}}$ and we found that the level two S-matrix, \mathbb{S}^{II} remains unchanged. In other words, this yields that the Bethe equations for any combination of bound states are given by:

$$\begin{aligned}
e^{ip_k L} &= \prod_{l=1, l \neq k}^{K^{\text{I}}} \left[S_0(p_k, p_l) \frac{x_k^+ - x_l^-}{x_k^- - x_l^+} \sqrt{\frac{x_l^+ x_k^-}{x_l^- x_k^+}} \right]^2 \prod_{\alpha=1}^2 \prod_{l=1}^{K_{(\alpha)}^{\text{II}}} \frac{x_k^- - y_l^{(\alpha)}}{x_k^+ - y_l^{(\alpha)}} \sqrt{\frac{x_k^+}{x_k^-}} \\
1 &= \prod_{l=1}^{K^{\text{I}}} \frac{y_k^{(\alpha)} - x_l^+}{y_k^{(\alpha)} - x_l^-} \sqrt{\frac{x_k^-}{x_k^+}} \prod_{l=1}^{K_{(\alpha)}^{\text{III}}} \frac{y_k^{(\alpha)} + \frac{1}{y_k^{(\alpha)}} - w_l^{(\alpha)} + \frac{i}{g}}{y_k^{(\alpha)} + \frac{1}{y_k^{(\alpha)}} - w_l^{(\alpha)} - \frac{i}{g}} \quad (7.94) \\
1 &= \prod_{l=1}^{K_{(\alpha)}^{\text{II}}} \frac{w_k^{(\alpha)} - y_k^{(\alpha)} - \frac{1}{y_k^{(\alpha)}} + \frac{i}{g}}{w_k^{(\alpha)} - y_k^{(\alpha)} - \frac{1}{y_k^{(\alpha)}} - \frac{i}{g}} \prod_{l \neq k}^{K_{(\alpha)}^{\text{III}}} \frac{w_k^{(\alpha)} - w_l^{(\alpha)} - \frac{2i}{g}}{w_k^{(\alpha)} - w_l^{(\alpha)} + \frac{2i}{g}},
\end{aligned}$$

with

$$x_k^+ + \frac{1}{x_k^+} - x_k^- - \frac{1}{x_k^-} = \frac{2i\ell_k}{g}, \quad \frac{x_k^+}{x_k^-} = e^{ip_k}. \quad (7.95)$$

However, note that apart from the parameters x^\pm , the phase factor $S_0(p_k, p_l)$ also depends on the bound states representation one considers (4.126).

7.4 Summary

In this chapter we derived the equations that capture the large volume spectrum of bound states of the $\text{AdS}_5 \times \text{S}^5$ superstring. They were derived by making use of the nested coordinate Bethe ansatz. In this procedure one makes a plane-wave type ansatz for the eigenstates of the Hamiltonian.

The coefficients in this ansatz depend crucially on the S-matrix of the system. The non-trivial matrix structure of the S-matrix leads to a nested structure which emerges in the form of a set of auxiliary momenta y_i, w_i . One then imposes periodic boundary conditions on the system which results in a set of coupled equations (7.94) that restrict the particle momenta. The solutions of these equations describe the large volume spectrum of $\text{AdS}_5 \times \text{S}^5$ bound states.

Algebraic Bethe Ansatz

Apart from the coordinate Bethe ansatz, there exists also another method of diagonalizing an integrable Hamiltonian. This method goes under the name of algebraic Bethe ansatz [50, 149]. The key feature in this method is the transfer matrix, which is a generator of conserved charges. The eigenvalues of the transfer matrix are also important since they encode the asymptotic behavior of the TBA equations. In the algebraic Bethe ansatz one constructs all eigenvalues of the transfer matrix by first defining a vacuum eigenstate and then introducing creation operators which generate excited eigenstates.

In this chapter we will apply this procedure to the AdS/CFT problem at hand. We will first give the necessary definitions of monodromy and transfer matrices and introduce the vacuum and find its eigenvalue. After this we will define the creation operators and build the excited states and the corresponding eigenvalues. From these one can again read off the nested Bethe equations that were derived in the previous chapter. Finally we discuss a fusion procedure which allows to derive the bound state transfer matrices from the case where all particles are in the fundamental representation.

8.1 Monodromy and transfer matrices

Consider K^1 bound state particles with bound state numbers $\ell_1, \dots, \ell_{K^1}$ and momenta p_1, \dots, p_{K^1} . To these particles we add an auxiliary one, with momentum q and bound state number ℓ_0 . Any state of this system lives in the following tensor product space

$$\mathcal{V} := V_{\ell_0}(q) \otimes V_{\ell_1}(p_1) \otimes \dots \otimes V_{\ell_{K^1}}(p_{K^1}), \quad (8.1)$$

where V_{ℓ_i} is the carrier space of the bound state representation with the number ℓ_i . We split the states in the space \mathcal{V} into an auxiliary part and a physical part:

$$|A\rangle_0 \otimes |B\rangle_{K^1} \in \mathcal{V},$$

where $|A\rangle_0 \in V_{\ell_0}(q)$ and¹ $|B\rangle_{K^1} \in V_P := \bigotimes_i V_{\ell_i}(p_i)$. The monodromy matrix acting in the space \mathcal{V} is defined as follows:

$$\mathcal{T}_{\ell_0}(q|\vec{p}) := \prod_{i=1}^{K^1} \mathbb{S}_{0k}(q, p_i), \quad (8.2)$$

where $\mathbb{S}_{0k}(q, p_k)$ is the bound state S-matrix describing scattering between the auxiliary particle, with momentum q and bound state number ℓ_0 , and a ‘physical’ particle, with momentum p_k and bound state number ℓ_k . For convenience we will work with the canonically normalized S-matrix, meaning that we take $\mathbb{S} w_1^{\ell_1} w_2^{\ell_2} = w_1^{\ell_1} w_2^{\ell_2}$. We will include the appropriate overall normalization S_0 (equation (4.126)) in the end.

The monodromy matrix can be seen as a $4\ell_0 \times 4\ell_0$ dimensional matrix in the auxiliary space $V_{\ell_0}(q)$, the corresponding matrix elements being themselves operators on V_P . Indeed, introducing a basis $|e_I\rangle$ for $V_{\ell_0}(q)$, with the index I labelling a $4\ell_0$ -dimensional space, and a basis $|f_A\rangle$ for V_P , the action of the monodromy matrix $\mathcal{T} \equiv \mathcal{T}_{\ell_0}(q|\vec{p})$ on the total space \mathcal{V} can be written as

$$\mathcal{T}(|e_I\rangle \otimes |f_A\rangle) = \sum_{J,B} T_{IA}^{JB} |e_J\rangle \otimes |f_B\rangle. \quad (8.3)$$

The matrix entries of the monodromy matrix can then be denoted as

$$\mathcal{T}|e_I\rangle = \sum_J \mathcal{T}_I^J |e_J\rangle, \quad (8.4)$$

while the action of the matrix elements \mathcal{T}_I^J as operators on V_P can easily be read off:

$$\mathcal{T}_I^J |f_A\rangle = \sum_B T_{IA}^{JB} |f_B\rangle. \quad (8.5)$$

The operators \mathcal{T}_I^J have non-trivial commutation relations among themselves. Consider two different auxiliary spaces $V_{\ell_0}(q), V_{\tilde{\ell}_0}(\tilde{q})$. The Yang-Baxter equation for \mathbb{S} implies that

$$\mathbb{S}(q, \tilde{q}) \mathcal{T}_{\ell_0}(q|\vec{p}) \mathcal{T}_{\tilde{\ell}_0}(\tilde{q}|\vec{p}) = \mathcal{T}_{\tilde{\ell}_0}(\tilde{q}|\vec{p}) \mathcal{T}_{\ell_0}(q|\vec{p}) \mathbb{S}(q, \tilde{q}), \quad (8.6)$$

where $\mathbb{S}(q, \tilde{q})$ is the S-matrix describing the scattering between two bound state particles with bound state numbers $\ell_0, \tilde{\ell}_0$ and momenta q, \tilde{q} respectively. By explicitly working out these relations, one finds the commutation relations between the different matrix elements of the monodromy matrix. The fundamental commutation relations (8.6) constitute the cornerstone of the Algebraic Bethe Ansatz [50].

¹All the tensor products are defined with increasing order of the index as $1, 2, \dots, \max$.

It is convenient to pick up the following explicit basis $|e_I\rangle$ in the space $V_{\ell_0}(q)$

$$\begin{aligned} e_{\alpha;k} &:= \theta_{\alpha} w_1^{\ell_0-k-1} w_2^k, \\ e_k &:= w_1^{\ell_0-k} w_2^k, \\ e_{34;k} &:= \theta_3 \theta_4 w_1^{\ell_0-k-1} w_2^{k-1}. \end{aligned} \tag{8.7}$$

The transfer matrix is then defined as

$$\mathcal{T}_0(q|\vec{p}) := \text{str}_0 \mathcal{T}_{\ell_0}(q|\vec{p}), \tag{8.8}$$

and it can be viewed as an operator acting on the physical space V_P . In terms of the operator entries of the monodromy matrix, the transfer matrix is written as

$$\mathcal{T}_0(q|\vec{p}) = \sum_{k=0}^{\ell_0} \mathcal{T}_k^k + \sum_{k=1}^{\ell_0-1} \mathcal{T}_{34;k}^{34;k} - \sum_{k=0}^{\ell_0-1} \sum_{\alpha=3,4} \mathcal{T}_{\alpha;k}^{\alpha;k}. \tag{8.9}$$

Let us now briefly indicate why the transfer matrix is important in integrable models. From (8.6) one can easily deduce that

$$[\mathcal{T}_0(q|\vec{p}), \mathcal{T}_0(\tilde{q}|\vec{p})] = 0. \tag{8.10}$$

Hence by expanding $\mathcal{T}_0(q|\vec{p})$ in q one generates an infinite set of commuting charges.

In the remainder of this paper we will study the action of $\mathcal{T}_0(q|\vec{p})$ on the physical space V_P in detail and derive its eigenvalues.

8.2 Diagonalization of the transfer matrix

We start by defining a vacuum state

$$|0\rangle_P = w_1^{\ell_1} \otimes \dots \otimes w_1^{\ell_{K^1}}. \tag{8.11}$$

We then compute the action of the transfer matrix on this state, which appears to be one of its eigenstates, and afterwards use specific elements of the monodromy matrix to generate the whole spectrum of eigenvalues. Imposing the eigenstate condition should result in the determination of the full set of eigenvalues and associated Bethe equations, therefore providing the complete solution of the asymptotic spectral problem.

8.2.1 Eigenvalue of the transfer matrix on the vacuum

As promised, we first deduce the action of the transfer matrix on the vacuum. We will do this for each of the separate sums in (8.9). Let us start with the fermionic part, *i.e.* we want to compute

$$\sum_{k=0}^{\ell_0-1} \mathcal{T}_{\alpha;k}^{\alpha;k} |0\rangle_P, \quad \alpha = 3, 4. \tag{8.12}$$

Taking into account the explicit form of the S-matrix elements entering the monodromy matrix, we find that the only contribution to $\mathcal{T}_{\alpha;k}^{\alpha;k}|0\rangle$ comes from diagonal scattering elements. To be precise, one finds

$$\mathcal{T}_{\alpha;k}^{\alpha;k}|0\rangle_P = \prod_i \mathcal{Y}_{k;1}^{k,0;1}(q, p_i)|0\rangle_P, \quad (8.13)$$

where $\mathcal{Y}_{k;1}^{k,0;1}(q, p_i)$ are Case II S-matrix elements (see section 4.8 for explicit expressions). By explicitly working out this expression, one finds

$$\mathcal{Y}_{k;1}^{k,0;1}(q, p_i) = \frac{x_0^+ - x_i^+}{x_0^- - x_i^+} \sqrt{\frac{x_0^-}{x_0^+}} \left[1 - \frac{k}{u_0 - u_i + \frac{\ell_0 - \ell_i}{2}} \right] \mathcal{X}_k^{k,0}(q, p_i), \quad (8.14)$$

where x_0^\pm are defined in terms of the momentum q as in (3.12), and one uses equation (4.71):

$$\begin{aligned} \mathcal{X}_k^{k,0}(q, p_i) &= \mathcal{D} \frac{\prod_{j=0}^{k-1} u_0 - u_i + \frac{\ell_0 - \ell_i - 2j}{2}}{\prod_{j=1}^k u_0 - u_i + \frac{\ell_0 + \ell_i - 2j}{2}} \quad k = 1, \dots, \ell_0 - 1, \\ \mathcal{X}_0^{0,0}(q, p_i) &= \mathcal{D} = \frac{x_0^- - x_i^+}{x_0^+ - x_i^-} \sqrt{\frac{x_0^+}{x_0^-} \frac{x_i^-}{x_i^+}}. \end{aligned} \quad (8.15)$$

Obviously, the contribution of $\mathcal{T}_{\alpha;k}^{\alpha;k}$ is the same for $\alpha = 3, 4$. Here x_m^\pm , with $m = 0, 1, \dots, K^I$, are the constrained parameters (λ is the 't Hooft coupling)

$$x_m^+ + \frac{1}{x_m^+} - x_m^- - \frac{1}{x_m^-} = 2\ell_m \frac{i}{g}, \quad g = \frac{\sqrt{\lambda}}{2\pi}$$

related to the particle momenta as $p_m = \frac{1}{i} \log \frac{x_m^+}{x_m^-}$, cf. equation (3.12). Also, u_m represents the corresponding rapidity variable given by (see equation (3.77))

$$x_m^\pm + \frac{1}{x_m^\pm} = \frac{2i}{g} u_m \pm \frac{i}{g} \ell_m. \quad (8.16)$$

Next, we consider the more involved bosonic part. This can be written as

$$\mathcal{T}_0^0 + \mathcal{T}_{\ell_0}^{\ell_0} + \sum_{k=1}^{\ell_0-1} \left\{ \mathcal{T}_k^k + \mathcal{T}_{34;k}^{34;k} \right\}. \quad (8.17)$$

We first determine $\mathcal{T}_0^0|0\rangle_P$ and $\mathcal{T}_{\ell_0}^{\ell_0}|0\rangle_P$. For these operators, one again finds that only diagonal scattering elements of the S-matrices contribute, which leads to

$$\begin{aligned} \mathcal{T}_0^0|0\rangle_P &= \prod_i \mathcal{Z}_{0;1}^{0,0;1}(q, p_i) |0\rangle_P, \\ \mathcal{T}_{\ell_0}^{\ell_0}|0\rangle_P &= \prod_i \mathcal{Z}_{\ell_0;1}^{\ell_0,0;1}(q, p_i) |0\rangle_P. \end{aligned} \quad (8.18)$$

These matrix elements can be computed explicitly and give

$$\mathcal{T}_0^0|0\rangle_P = |0\rangle_P, \quad (8.19)$$

$$\mathcal{T}_{\ell_0}^{\ell_0}|0\rangle_P = \left\{ \prod_{i=1}^{K^1} \frac{(x_0^- - x_i^-)(1 - x_0^- x_i^+)}{(x_0^- - x_i^+)(1 - x_0^+ x_i^+)} \sqrt{\frac{x_0^+ x_i^+}{x_0^- x_i^-}} \mathcal{X}_{\ell_0}^{\ell_0,0}(q, p_i) \right\} |0\rangle_P. \quad (8.20)$$

where we define

$$\mathcal{X}_{\ell_0}^{\ell_0,0}(q, p_i) = \mathcal{D} \frac{\prod_{j=0}^{\ell_0-1} u_0 - u_i + \frac{\ell_0 - \ell_i - 2j}{2}}{\prod_{j=1}^{\ell_0} u_0 - u_i + \frac{\ell_0 + \ell_i - 2j}{2}}. \quad (8.21)$$

The next thing to consider is the sum

$$\sum_{k=1}^{\ell_0-1} \left\{ \mathcal{T}_k^k + \mathcal{T}_{34;k}^{34;k} \right\}. \quad (8.22)$$

While in the previous computations one could simply restrict to the diagonal elements, one obtains instead a matrix structure for this last piece. This is due to the fact that there are scattering processes that relate $w_2 \leftrightarrow \theta_\alpha \theta_\beta$. To be more precise, for the action of \mathcal{T}_k^k and $\mathcal{T}_{34,k}^{34,k}$ one finds

$$\mathcal{T}_k^k|0\rangle_P = \sum_{a_1 \dots a_{K^1}=1,3} \mathcal{Z}_{k;1}^{k,0;a_1}(q, p_1) \mathcal{Z}_{k;a_1}^{k,0;a_2}(q, p_2) \dots \mathcal{Z}_{k;a_{K^1}}^{k,0;1}(q, p_{K^1})|0\rangle_P, \quad (8.23)$$

$$\mathcal{T}_{34,k}^{34,k}|0\rangle_P = \sum_{a_1 \dots a_{K^1}=1,3} \mathcal{Z}_{k;3}^{k,0;a_1}(q, p_1) \mathcal{Z}_{k;a_1}^{k,0;a_2}(q, p_2) \dots \mathcal{Z}_{k;a_{K^1}}^{k,0;3}(q, p_{K^1})|0\rangle_P. \quad (8.24)$$

In order to evaluate the above expressions explicitly, it proves useful to use a slightly more general reformulation². One can reintroduce the elements $\mathcal{T}_k^{34,k}$ and $\mathcal{T}_{34,k}^k$ from the monodromy matrix. Their action on the vacuum is

$$\mathcal{T}_k^{34,k}|0\rangle_P = \sum_{a_1 \dots a_{K^1}=1,3} \mathcal{Z}_{k;1}^{k,0;a_1}(q, p_1) \mathcal{Z}_{k;a_1}^{k,0;a_2}(q, p_2) \dots \mathcal{Z}_{k;a_{K^1}}^{k,0;3}(q, p_{K^1})|0\rangle_P, \quad (8.25)$$

$$\mathcal{T}_{34,k}^k|0\rangle_P = \sum_{a_1 \dots a_{K^1}=1,3} \mathcal{Z}_{k;3}^{k,0;a_1}(q, p_1) \mathcal{Z}_{k;a_1}^{k,0;a_2}(q, p_2) \dots \mathcal{Z}_{k;a_{K^1}}^{k,0;1}(q, p_{K^1})|0\rangle_P. \quad (8.26)$$

They describe the mixing between the states $|e_{34,k}\rangle$ and $|e_k\rangle$. If we consider the two-dimensional vector space spanned by $|e_{34,k}\rangle$ and $|e_k\rangle$ for fixed $k \in \{1, \dots, \ell_0 - 1\}$, we see that the above elements define a 2×2 dimensional matrix

$$\mathcal{T}_{2 \times 2} = \begin{pmatrix} \mathcal{T}_k^k & \mathcal{T}_k^{34,k} \\ \mathcal{T}_{34,k}^k & \mathcal{T}_{34,k}^{34,k} \end{pmatrix}, \quad (8.27)$$

and the bosonic part, $\mathcal{T}_k^k + \mathcal{T}_{34,k}^{34,k}$, of the transfer matrix is just the trace of this matrix. Moreover, it is easily seen from the definition of the transfer matrix that this matrix factorizes

$$\mathcal{T}_{2 \times 2} = \prod_{i=1}^K \begin{pmatrix} \mathcal{Z}_{k;1}^{k,0;1}(q, p_i) & \mathcal{Z}_{k;1}^{k,0;3}(q, p_i) \\ \mathcal{Z}_{k;3}^{k,0;1}(q, p_i) & \mathcal{Z}_{k;3}^{k,0;3}(q, p_i) \end{pmatrix}. \quad (8.28)$$

²We remark that this computation has been performed at weak coupling in [63, 64].

The trace of this matrix is given by the sum of its eigenvalues, hence it remains to find the eigenvalues of this matrix. Actually, it is easily checked that the eigenvectors of

$$\begin{pmatrix} \mathcal{Z}_{k;1}^{k,0;1}(q, p_i) & \mathcal{Z}_{k;1}^{k,0;3}(q, p_i) \\ \mathcal{Z}_{k;3}^{k,0;1}(q, p_i) & \mathcal{Z}_{k;3}^{k,0;3}(q, p_i) \end{pmatrix} \quad (8.29)$$

are independent of p_i . In other words, these are automatically eigenvectors of $\mathcal{T}_{2 \times 2}$, and the corresponding eigenvalues are the product of the eigenvalues of the above matrices. The individual eigenvalues are given by

$$\begin{aligned} \lambda_{\pm}(q, p_i, k) = & \frac{\mathcal{Z}_k^{k,0}}{2\mathcal{D}} \left[1 - \frac{(x_i^- x_0^+ - 1)(x_0^+ - x_i^+)}{(x_i^- - x_0^+)(x_0^+ x_i^+ - 1)} + \frac{2ik}{g} \frac{x_0^+(x_i^- + x_i^+)}{(x_i^- - x_0^+)(x_0^+ x_i^+ - 1)} \right. \\ & \left. \pm \frac{ix_0^+(x_i^- - x_i^+)}{(x_i^- - x_0^+)(x_0^+ x_i^+ - 1)} \sqrt{\left(\frac{2k}{g}\right)^2 + 2i \left[x_0^+ + \frac{1}{x_0^+}\right] \frac{2k}{g} - \left[x_0^+ - \frac{1}{x_0^+}\right]^2} \right]. \end{aligned} \quad (8.30)$$

The action of the transfer matrix on the vacuum is now given by the summing of all the above terms (8.14, 8.19, 8.20, 8.30). From this it is easily seen that the vacuum is indeed an eigenvector of the transfer matrix with the following eigenvalue

$$\begin{aligned} \Lambda(q|\vec{p}) = & 1 + \prod_{i=1}^{K^1} \left[\frac{(x_0^- - x_i^-)(1 - x_0^- x_i^+)}{(x_0^- - x_i^+)(1 - x_0^+ x_i^+)} \sqrt{\frac{x_0^+ x_i^+}{x_0^- x_i^-}} \mathcal{Z}_{\ell_0,0}^{\ell_0,0} \right] \\ & - 2 \sum_{k=0}^{\ell_0-1} \prod_{i=1}^{K^1} \left(\frac{x_0^+ - x_i^+}{x_0^- - x_i^+} \sqrt{\frac{x_0^-}{x_0^+}} \left[1 - \frac{k}{u_0 - u_i + \frac{\ell_0 - \ell_i}{2}} \right] \mathcal{Z}_k^{k,0} \right) \\ & + \sum_{k=1}^{\ell_0-1} \prod_{i=1}^{K^1} \lambda_+(q, p_i, k) + \sum_{k=1}^{\ell_0-1} \prod_{i=1}^{K^1} \lambda_-(q, p_i, k). \end{aligned} \quad (8.31)$$

For the fundamental case ($\ell_0 = \ell_i = 1 \forall i$), this reduces to

$$\begin{aligned} \mathcal{T}_0(q|\vec{p})|0\rangle_P = & \left\{ \prod_i \mathcal{Z}_{0;1}^{0,0;1}(q, p_i) + \prod_i \mathcal{Z}_{1;1}^{1,0;1}(q, p_i) - 2 \prod_i \mathcal{Z}_{0;1}^{0,0;1}(q, p_i) \right\} |0\rangle_P \\ = & \left\{ 1 + \prod_{i=1}^{K^1} \frac{1 - \frac{1}{x_0^- x_i^+} x_0^+ - x_i^+}{1 - \frac{1}{x_0^- x_i^-} x_0^+ - x_i^-} - 2 \prod_{i=1}^{K^1} \frac{x_0^+ - x_i^+}{x_0^+ - x_i^-} \sqrt{\frac{x_i^-}{x_i^+}} \right\} |0\rangle_P. \end{aligned} \quad (8.32)$$

We would like to point out that the square roots in the eigenvalues λ_{\pm} never appear in the vacuum eigenvalue. This is because the square root part only depends on the auxiliary momentum q , and it can be seen that, after summing the contribution from λ_+ and λ_- , only even powers of this square root piece survive.

8.2.2 Creation operators and excited states

The next step in the algebraic Bethe ansatz is to introduce creation operators. These operators will be certain entries from our monodromy matrix. By acting with these operators on the

vacuum one creates new (excited) states, which, under conditions to be determined, will be eigenstates of the transfer matrix. We will need to specify which monodromy matrix entries correspond to creation operators for our purposes.

Recall that, from the symmetry invariance of the S-matrix, one can deduce that the quantum numbers K^{II} (total number of fermions) and K^{III} (total number of fermions of a definite species, say, 3) are conserved upon acting with the monodromy matrix on a state (8.2). Any element \mathcal{T}_I^J is called a creation operator if $K^{\text{II}}(|e_I\rangle_0) > K^{\text{II}}(|e_J\rangle_0)$, it is called an annihilation operator if $K^{\text{II}}(|e_I\rangle_0) < K^{\text{II}}(|e_J\rangle_0)$ and diagonal if $K^{\text{II}}(|e_I\rangle_0) = K^{\text{II}}(|e_J\rangle_0)$. The reason for this assignment is the following. Consider a creation operator \mathcal{T}_I^J and any physical state $|A\rangle_P$. The action of a creation operator is defined via (8.3). Since the total number K^{II} is preserved, and the K^{II} charge in the auxiliary space has decreased, it has necessarily increased in the physical space. The number K^{II} corresponds to the number of fermions in the system, hence, by acting with \mathcal{T}_I^J on $|A\rangle_P$, one creates extra fermions in the physical space. Notice that this also implies that acting with an annihilation operator on the vacuum gives zero, whence the name.

We will create excited states by considering auxiliary fundamental representations with momenta λ_i . We will use these to define creation operators $B_\alpha(\lambda_i), F(\lambda_i)$. Since these representations are fundamental, their monodromy matrices are only 4×4 -dimensional. Our discussion will be very similar to the treatment of the algebraic Bethe ansatz for the Hubbard model which was first performed in [150, 151]. In order to make contact with the treatment of [48] and with the standard notation used for the Hubbard model, we parameterize this monodromy matrix as

$$\begin{pmatrix} B & B_3 & B_4 & F \\ C_3 & A_3^3 & A_4^3 & B_3^* \\ C_4 & A_3^4 & A_4^4 & B_4^* \\ C & C_3^* & C_4^* & D \end{pmatrix}. \quad (8.33)$$

One finds two seemingly different sets of creation operators $B_3(\lambda_i), B_4(\lambda_i), F(\lambda_i)$ and $B_3^*(\lambda_i), B_4^*(\lambda_i), F(\lambda_i)$. As discussed in [150], it is enough to restrict to one set. In what follows, we will use the operators $B_3(\lambda_i), B_4(\lambda_i), F(\lambda_i)$ to create fermionic excitations out of the vacuum.

A generic excited state will now be formed by acting with a number of those operators on the vacuum, e.g. one can consider states like

$$B_3(\lambda_1)B_4(\lambda_2)|0\rangle. \quad (8.34)$$

To find out whether this is an eigenstate of the transfer matrix, one has to commute the diagonal elements of the transfer matrices through the creation operators and let them act on the vacuum. Imposing the eigenstate condition will in general give constraints on the momenta λ_i . The explicit commutation relations will be the subject of the next section.

8.2.3 Commutation relations

In order to compute the action of the transfer matrix on an excited state, we need to compute the commutation relations between the diagonal elements \mathcal{T}_A^A and the aforementioned creation operators. While we have to use creation operators in a fundamental auxiliary representation, the diagonal elements are to be taken in the bound state representation with generic ℓ_0 . The commutation relations follow from (8.6). We will report the complete derivation of one specific commutation relation, and only give the final result for the remaining ones. In the derivation, one has to pay particular attention to the fermionic nature of the operators.

Consider the operator $B_3(\lambda)$ and the element $\mathcal{T}_{3,k}^{3,k}$ from the transfer matrix. From (8.6), one finds

$$\mathbb{P}_{3,k|0}\mathbb{S}(q, \lambda)\mathcal{T}(q)\mathcal{T}(\lambda)e_{3,k}\tilde{e}_{3,0} = \mathbb{P}_{3,k|0}\mathcal{T}(\lambda)\mathcal{T}(q)\mathbb{S}(q, \lambda)e_{3,k}\tilde{e}_{3,0}, \quad (8.35)$$

where we have dropped the index ℓ_0 and chosen $\tilde{\ell}_0 = 1$, and where the tilde on $\tilde{e}_{3,0}$ denotes a basis element in the second auxiliary space. The operator $\mathbb{P}_{A|B}$ is the projection operator onto the subspace generated by the basis element $e_A\tilde{e}_B$. The right hand side of the above equation gives

$$\begin{aligned} \mathbb{P}_{3,k|0}\mathcal{T}(\lambda)\mathcal{T}(q)\mathbb{S}(q, \lambda)e_{3,k}\tilde{e}_{3,0} &= \mathbb{P}_{3,k|0}\mathcal{X}_k^{k,0}\mathcal{T}(\lambda)\mathcal{T}(q)e_{3,k}\tilde{e}_{3,0} \\ &= \mathbb{P}_{3,k|0}\sum_{A,B}\mathcal{X}_k^{k,0}(-1)^{FA}(\mathcal{T}_3^B\tilde{e}_B)(\lambda)(\mathcal{T}_{3,k}^A(q)e_A) \\ &= \mathcal{X}_k^{k,0}(-1)^{F(3,k)}(\mathcal{T}_3^0\tilde{e}_0)(\lambda)(\mathcal{T}_{3,k}^{3,k}(q)e_{3,k}) \\ &= -\mathcal{X}_k^{k,0}B_3^0(\lambda)\mathcal{T}_{3,k}^{3,k}(q)e_{3,k}\tilde{e}_0. \end{aligned} \quad (8.36)$$

The left hand side reduces to

$$\begin{aligned} \mathbb{P}_{3,k|0}\mathbb{S}(q, \lambda)\mathcal{T}(q)\mathcal{T}(\lambda)e_{3,k}\tilde{e}_{3,0} &= -\mathbb{P}_{3,k|0}\mathbb{S}(q, \lambda)\mathcal{T}(q)(\mathcal{T}(\lambda)_3^B\tilde{e}_B)e_{3,k} \\ &= -\mathbb{P}_{3,k|0}\mathbb{S}(q, \lambda)\mathcal{T}(q)\{\mathcal{T}_3^0(\lambda)\tilde{e}_0 + \mathcal{T}_3^3(\lambda)\tilde{e}_3 + \mathcal{T}_3^1(\lambda)\tilde{e}_1\}e_{3,k} \\ &= \mathbb{P}_{3,k|0}\mathbb{S}(q, \lambda)(\mathcal{T}_{3,k}^A(q)e_A)\{\mathcal{T}_3^0(\lambda)\tilde{e}_0 + \mathcal{T}_3^3(\lambda)\tilde{e}_3 + \mathcal{T}_3^1(\lambda)\tilde{e}_1\}. \end{aligned} \quad (8.37)$$

Because of the projection, we only need to take into account terms that are mapped onto $e_{3,k}\tilde{e}_0$ by the action of the S-matrix. These are given by

$$\begin{aligned} \mathbb{P}_{3,k|0}\mathbb{S}(q, \lambda)\left\{\mathcal{T}_{3,k}^{3,k}(q)e_{3,k}\mathcal{T}_3^0(\lambda)\tilde{e}_0 + \mathcal{T}_{3,k}^{3,k-1}(q)e_{3,k-1}\mathcal{T}_3^1(\lambda)\tilde{e}_1 + \mathcal{T}_{3,k}^k(q)e_k\mathcal{T}_3^3(\lambda)\tilde{e}_{3,0}\right\} = \\ \mathbb{P}_{3,k|0}\mathbb{S}(q, \lambda)\left\{-\mathcal{T}_{3,k}^{3,k}(q)\mathcal{T}_3^0(\lambda)e_{3,k}\tilde{e}_0 - \mathcal{T}_{3,k}^{3,k-1}(q)\mathcal{T}_3^1(\lambda)e_{3,k-1}\tilde{e}_1 + \right. \\ \left. + \mathcal{T}_{3,k}^k(q)\mathcal{T}_3^3(\lambda)e_k\tilde{e}_{3,0} + \mathcal{T}_{3,k}^{34,k-1}(q)\mathcal{T}_3^3(\lambda)e_{34,k-1}\tilde{e}_{3,0}\right\}. \end{aligned} \quad (8.38)$$

Working this out explicitly yields

$$\begin{aligned} \mathbb{P}_{3,k|0}\mathbb{S}(q, \lambda) \left\{ \mathcal{T}_{3,k}^{3,k}(q)e_{3,k}\mathcal{T}_3^0(\lambda)\tilde{e}_0 + \mathcal{T}_{3,k}^{3,k-1}(q)e_{3,k-1}\mathcal{T}_3^1(\lambda)\tilde{e}_1 + \mathcal{T}_{3,k}^k(q)e_k\mathcal{T}_3^3(\lambda)\tilde{e}_{3,0} \right\} = \\ \left\{ -\mathcal{T}_{3,k}^{3,k}(q)\mathcal{T}_3^0(\lambda)\mathcal{Y}_{k;1}^{k,0;1} - \mathcal{Y}_{k;1}^{k-1,1;1}\mathcal{T}_{3,k}^{3,k-1}(q)\mathcal{T}_3^1(\lambda) + \right. \\ \left. + \mathcal{Y}_{k;2}^{k,1;1}\mathcal{T}_{3,k}^k(q)\mathcal{T}_3^3(\lambda) + \mathcal{Y}_{k;4}^{k,1;1}\mathcal{T}_{3,k}^{34,k-1}(q)\mathcal{T}_3^3(\lambda) \right\} e_{3,k}\tilde{e}_0. \end{aligned} \quad (8.39)$$

From this we now read off the final commutation relation³

$$\begin{aligned} \mathcal{X}_k^{k,0}B_3(\lambda)\mathcal{T}_{3,k}^{3,k}(q) &= \mathcal{Y}_{k;1}^{k,0;1}\mathcal{T}_{3,k}^{3,k}(q)B_3(\lambda) + \mathcal{Y}_{k;1}^{k-1,1;1}\mathcal{T}_{3,k}^{3,k-1}(q)C_3^*(\lambda) + \\ &\quad - \mathcal{Y}_{k;2}^{k,1;1}\mathcal{T}_{3,k}^k(q)A_3^3(\lambda) - \mathcal{Y}_{k;4}^{k,1;1}\mathcal{T}_{3,k}^{34,k-1}(q)A_3^3(\lambda). \end{aligned} \quad (8.40)$$

Notice that in the above relation the operators are ordered in such a way that all annihilation and diagonal elements are on the right. This is done because the action of those elements on the vacuum is known. We would also like to compare these commutation relations with [150, 151] for the Hubbard model. We see that the first and third term are also present in the Hubbard model. However, due to the fact that we are dealing with bound state representation, we also obtain *two additional terms*.

Generically, the commutation relations produce “wanted” terms, which are those which directly contribute to the eigenvalue, and other “unwanted” terms. The latter terms are those which need to vanish, in order for the state of our ansatz to be an eigenstate. In (8.40), one can easily see by acting on the vacuum that the wanted term is the first term on the right hand side, while the other terms are unwanted. The cancellation of the unwanted terms will give rise to certain constraints, which are precisely the auxiliary Bethe equations.

The other commutation relations one needs to compute are those with $\mathcal{T}_k^k, \mathcal{T}_{34,k}^{34,k}$ and $\mathcal{T}_{4,k}^{4,k}$. Their derivation is considerably more involved, especially the procedure of reordering them according to the above “annihilation and diagonal to the right” prescription. We will present the commutation relations we will actually need in the coming sections. We will give the wanted terms, and focus on one specific type of unwanted terms. Schematically, we will focus on the

³Throughout the rest of this section 3.3, if not otherwise indicated, the coefficient functions appearing have to be understood as $\mathcal{X} \equiv \mathcal{X}(q, \lambda)$, $\mathcal{Y} \equiv \mathcal{Y}(q, \lambda)$, $\mathcal{Z} \equiv \mathcal{Z}(q, \lambda)$ (indices are omitted here for simplicity).

following structure:

$$\left[\mathcal{T}_k^k(q) + \mathcal{T}_{\alpha\beta,k}^{\alpha\beta,k}(q) \right] B_\alpha(\lambda) = \frac{\mathcal{X}_k^{k,0}}{\mathcal{Y}_{k,0;1}^{k,0;1}} B_\alpha(\lambda) \left[\mathcal{T}_k^k(q) + \mathcal{T}_{\alpha\beta,k}^{\alpha\beta,k}(q) \right] + \quad (8.41)$$

$$+ \frac{\mathcal{Y}_{k,0;1}^{k,0;1}}{\mathcal{Y}_{k,0;1}^{k,0;1}} \mathcal{T}_{\alpha,k}^k(q) B(\lambda) + \dots$$

$$\begin{aligned} \mathcal{T}_{\alpha_1,k}^{\gamma,k}(q) B_{\alpha_2}(\lambda) &= \frac{\mathcal{X}_k^{k,0}}{\mathcal{Y}_{k,0;1}^{k,0;1}} B_{\beta_2}(\lambda) \mathcal{T}_{\beta_1,k}^{\gamma,k}(q) r_{\alpha_1\alpha_2}^{\beta_1\beta_2}(u_0 + \frac{\ell_0-1}{2} - k, u_\lambda) + \\ &+ \frac{\mathcal{Y}_{k,0;1}^{k,0;1}}{\mathcal{Y}_{k,0;1}^{k,0;1}} \mathcal{T}_{\beta_1,k}^k(q) A_{\beta_2}^\gamma(\lambda) r_{\alpha_1\alpha_2}^{\beta_1\beta_2}(u_\lambda, u_\lambda) + \dots \end{aligned} \quad (8.42)$$

Here, u_λ is given by (cf. (8.16))

$$u_\lambda = \frac{g}{2i} \left(x^+(\lambda) + \frac{1}{x^+(\lambda)} - \frac{i}{g} \right)$$

and $r_{\alpha\beta}^{\gamma\delta}(u_\lambda, u_\mu)$ are the components of the 6-vertex model S-matrix (8.93) with $U = -1$. We would like to point out that, when comparing this structure against formulas (34-36) of [150], one immediately recognizes a similarity between the commutation relations. As was shown in [48], for the case in which all representations are taken to be fundamental, the commutation relations do agree. The additional contributions coming from the fact that we are dealing with bound states in e.g. (8.40), will only generate a new class of unwanted terms. Hence, these new terms will not contribute to the eigenvalues.

Let us mention one commutation relation which is particularly straightforward to derive, namely, the one between two fermionic creation operators, as found from (8.6) with $\ell_0 = \tilde{\ell}_0 = 1$. This relation reads

$$\begin{aligned} B_\alpha(\lambda) B_\beta(\mu) &= -\mathcal{X}_0^{0,0}(\lambda, \mu) B_\delta(\mu) B_\gamma(\lambda) r_{\alpha\beta}^{\gamma\delta}(u_\lambda, u_\mu) \\ &+ \frac{\mathcal{X}_{1;6}^{1,0;1}(\lambda, \mu)}{\mathcal{X}_{1;1}^{1,0;1}(\lambda, \mu)} [F(\lambda) B(\mu) - F(\mu) B(\lambda)] \epsilon_{\alpha\beta}. \end{aligned} \quad (8.43)$$

This reproduces the result of [48], and, in this way, one can see the emergence of nesting. As a matter of fact, in [48, 150, 151] the appearance of the 6-vertex model S-matrix was used to completely fix the form of the excited eigenstates, and this can also be done in our case.

8.2.4 First excited state

The first excited state is of the form

$$|1\rangle = \mathcal{F}^\alpha B_\alpha(\lambda) |0\rangle_P, \quad (8.44)$$

where we sum over the repeated fermionic index. This state has $K^\Pi = 1$. As previously discussed, all the commutation relations are ordered in such a way that all annihilation and diagonal operators are on the right. From the commutation relations (8.41) one finds

$$\begin{aligned} \left[\mathcal{T}_k^k(q) + \mathcal{T}_{\alpha\beta,k}^{\alpha\beta,k}(q) \right] \mathcal{F}^\alpha B_\alpha(\lambda) |0\rangle_P &= \frac{\mathcal{X}_k^{k,0}}{\mathcal{Y}_{k;1}^{k,0;1}} \mathcal{F}^\alpha B_\alpha(\lambda) \left[\mathcal{T}_k^k(q) + \mathcal{T}_{\alpha\beta,k}^{\alpha\beta,k}(q) \right] |0\rangle_P \\ &\quad + \frac{\mathcal{Y}_{k;2}^{k,0;1}}{\mathcal{Y}_{k;1}^{k,0;1}} \mathcal{F}^\alpha \mathcal{T}_{\alpha,k}^k(q) B(\lambda) |0\rangle_P + \dots, \end{aligned} \quad (8.45)$$

$$\begin{aligned} \left[\mathcal{T}_{\alpha_1,k}^{\alpha_1,k}(q) \right] \mathcal{F}^{\alpha_2} B_{\alpha_2}(\lambda) |0\rangle_P &= \frac{\mathcal{X}_k^{k,0}}{\mathcal{Y}_{k;1}^{k,0;1}} \mathcal{F}^{\alpha_2} r_{\alpha_1\alpha_2}^{\beta_1\beta_2}(u_0 + \frac{\ell_0-1}{2} - k, u_\lambda) B_{\beta_2}(\lambda) \left[\mathcal{T}_{\alpha_1,k}^{\beta_1,k}(q) \right] |0\rangle_P \\ &\quad + \frac{\mathcal{Y}_{k;2}^{k,0;1}}{\mathcal{Y}_{k;1}^{k,0;1}} \mathcal{F}^{\alpha_2} r_{\alpha_1\alpha_2}^{\beta_1\beta_2}(u_\lambda, u_\lambda) \mathcal{T}_{\beta_2,k}^k(q) A_{\beta_1}^{\alpha_1}(\lambda) |0\rangle_P + \dots, \end{aligned} \quad (8.46)$$

where we remind that we concentrate on only one type of unwanted terms, for the sake of clarity. Notice the appearance of six-vertex model S-matrix r (see appendix A) in the commutation relations. The coefficient functions appearing in the above two formulas have to be understood as $\mathcal{X} \equiv \mathcal{X}(q, \lambda)$, $\mathcal{Y} = \mathcal{Y}(q, \lambda)$ (indices are omitted here for simplicity).

Since $\mathcal{T}_{\beta,k}^{\alpha,k} |0\rangle_P \sim \delta_\beta^\alpha |0\rangle_P$, we find that $|1\rangle$ can only be an eigenstate of the transfer matrix if

$$\mathcal{F}^\alpha r_{\gamma\alpha}^{\gamma\beta}(u_0 + \frac{\ell_0-1}{2} - k, u_\lambda) \sim \mathcal{F}^\beta. \quad (8.47)$$

This means that \mathcal{F}^α is an eigenvector of the transfer matrix of the 6-vertex model. Luckily, one finds that the eigenstates of the 6-vertex model are independent of the auxiliary momentum. The k dependence in the above r-matrix appears in the eigenvalue $\Lambda^{(6v)}$, where $\Lambda^{(6v)}$ is the eigenvalue of the auxiliary 6-vertex model. From (8.105) we find ($K = K^\Pi = 1$)

$$\Lambda^{(6v)}(u_0|u_\lambda) = \prod_{i=1}^{K^\text{III}} \frac{1}{b(w_i, u_0 + \frac{\ell_0-1}{2} - k)} + b(u_0 + \frac{\ell_0-1}{2} - k, u_\lambda) \prod_{i=1}^{K^\text{III}} \frac{1}{b(u_0 + \frac{\ell_0-1}{2} - k, w_i)},$$

together with the auxiliary equation (8.106)

$$b(w_j, u_\lambda) = \prod_{i=1, i \neq j}^{K^\text{III}} \frac{b(w_j, w_i)}{b(w_i, w_j)}. \quad (8.48)$$

We also have to deal with the unwanted terms. Here we remark that, since we have chosen \mathcal{F}^α to be an eigenvector of the 6-vertex S-matrix, this also affects the unwanted terms. One explicitly finds that the unwanted terms are proportional to

$$\left\{ \Lambda^{(6v)}(u_\lambda|u_\lambda) A_\alpha^\alpha(\lambda) - B(\lambda) \right\} |0\rangle_P. \quad (8.49)$$

Cancelling these unwanted terms thus leads us to the following auxiliary Bethe equations:

$$\prod_{i=1}^{K^\text{I}} \frac{x^+(\lambda) - x_i^-}{x^+(\lambda) - x_i^+} \sqrt{\frac{x_i^+}{x_i^-}} = \Lambda^{(6v)}(u_\lambda|u_\lambda). \quad (8.50)$$

In order to make contact with the bound state Bethe equations (7.94), let us define $y \equiv x^+(\lambda)$ and rescale $w \rightarrow \frac{g}{2i}w$. We find that $|1\rangle$ is an eigenstate, provided the auxiliary Bethe equations hold⁴:

$$\begin{aligned} \prod_{i=1}^{K^I} \frac{y - x_i^-}{y - x_i^+} \sqrt{\frac{x_i^+}{x_i^-}} &= \prod_{i=1}^{K^{III}} \frac{w_i - y - \frac{1}{y} - \frac{i}{g}}{w_i - y - \frac{1}{y} + \frac{i}{g}}, \\ \frac{w_i - y - \frac{1}{y} + \frac{i}{g}}{w_i - y - \frac{1}{y} - \frac{i}{g}} &= \prod_{j=1, j \neq i}^{K^{III}} \frac{w_i - w_j + \frac{2i}{g}}{w_i - w_j - \frac{2i}{g}}. \end{aligned} \quad (8.51)$$

This exactly matches with the auxiliary bound state Bethe ansatz equations (7.94) derived in the previous chapter. The corresponding eigenvalue is

$$\begin{aligned} \Lambda(q|\vec{p}) &= \frac{y-x_0^-}{y-x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} + \\ &+ \frac{y-x_0^-}{y-x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} \left[\frac{x_0^+ + \frac{1}{x_0^+} - y - \frac{1}{y}}{x_0^+ + \frac{1}{x_0^+} - y - \frac{1}{y} - \frac{2i\ell_0}{g}} \right] \prod_{i=1}^{K^I} \left[\frac{(x_0^- - x_i^-)(1 - x_0^- x_i^+)}{(x_0^- - x_i^+)(1 - x_0^+ x_i^-)} \sqrt{\frac{x_0^+ x_i^+}{x_0^- x_i^-}} \mathcal{X}_{\ell_0}^{\ell_0, 0} \right] \\ &+ \sum_{k=1}^{\ell_0-1} \frac{y-x_0^-}{y-x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} \left[\frac{x_0^+ + \frac{1}{x_0^+} - y - \frac{1}{y}}{x_0^+ + \frac{1}{x_0^+} - y - \frac{1}{y} - \frac{2ik}{g}} \right] \left\{ \prod_{i=1}^{K^I} \lambda_+(q, p_i, k) + \prod_{i=1}^{K^I} \lambda_-(q, p_i, k) \right\} \\ &- \sum_{k=0}^{\ell_0-1} \frac{y-x_0^-}{y-x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} \left[\frac{x_0^+ + \frac{1}{x_0^+} - y - \frac{1}{y}}{x_0^+ + \frac{1}{x_0^+} - y - \frac{1}{y} - \frac{2ik}{g}} \right] \prod_{i=1}^{K^I} \frac{x_0^+ - x_i^+}{x_0^- - x_i^-} \sqrt{\frac{x_0^-}{x_0^+}} \left[1 - \frac{k}{u_0 - u_i + \frac{\ell_0 - \ell_i}{2}} \right] \times \\ &\times \mathcal{X}_k^{k, 0} \left\{ \prod_{i=1}^{K^{III}} \frac{w_i - x_0^+ - \frac{1}{x_0^+} + \frac{i(2k-1)}{g}}{w_i - x_0^+ - \frac{1}{x_0^+} + \frac{i(2k+1)}{g}} + \frac{y + \frac{1}{y} - x_0^+ - \frac{1}{x_0^+} + \frac{2ik}{g}}{y + \frac{1}{y} - x_0^+ - \frac{1}{x_0^+} + \frac{2i(k+1)}{g}} \prod_{i=1}^{K^{III}} \frac{w_i - x_0^+ - \frac{1}{x_0^+} + \frac{i(2k+3)}{g}}{w_i - x_0^+ - \frac{1}{x_0^+} + \frac{i(2k+1)}{g}} \right\}. \end{aligned} \quad (8.52)$$

We stress once again that the above eigenvalue is for the canonically normalized S-matrix, i.e. it is normalized such that $\mathbb{S}w_1^{\ell_1}w_2^{\ell_2} = w_1^{\ell_1}w_2^{\ell_2}$. The dependence of Λ on the bound state numbers of the physical particles is hidden in the parameters x_i^\pm and in the S-matrix element \mathcal{X} . Notice that, when projected in the fundamental representation, the formula above reproduces the result of [48].

8.2.5 General result and Bethe equations

As was stressed before, by comparing our commutation relations against (34)-(36) from [150, 151], one immediately notices several similarities. It turns out that one can closely follow the derivation presented in those papers, and from the diagonal terms read off the general eigenvalue. Furthermore, cancelling the first few unwanted terms reveals itself as sufficient to derive the complete set of auxiliary Bethe equations.

⁴We remark that, for $K^{III} = 0$, the solution of (8.51) correspond to the highest weight state of the auxiliary six-vertex model, while, for $K^{III} = 1$, one formally obtains a solution only if some of the auxiliary roots are equal to infinity. This corresponds to a descendent of the highest weight state under the $\mathfrak{su}(2)$ symmetry.

More specifically, the results of appendix B and the previously known results for the case when all physical legs are in the fundamental representation indicate the generalization of the formula for the transfer-matrix eigenvalues to multiple excitations. In terms of S-matrix elements, this generalization is given by

$$\begin{aligned}
\Lambda(q|\vec{p}) = & \prod_{m=1}^{K^{\text{II}}} \frac{\mathcal{X}_0^{0,0}(q, \lambda_m)}{\mathcal{Y}_{0;1}^{0,0;1}(q, \lambda_m)} + \prod_{i=1}^{K^{\text{I}}} \mathcal{Z}_{\ell_0;1}^{\ell_0,0;1}(q, p_i) \prod_{m=1}^{K^{\text{II}}} \frac{\mathcal{X}_{\ell_0}^{\ell_0,0}(q, \lambda_m)}{\mathcal{Y}_{\ell_0;1}^{\ell_0,0;1}(q, \lambda_m)} + \\
& \sum_{k=1}^{\ell_0-1} \prod_{m=1}^{K^{\text{II}}} \frac{\mathcal{X}_k^{k,0}(q, \lambda_m)}{\mathcal{Y}_{k;1}^{k,0;1}(q, \lambda_m)} \left\{ \prod_{i=1}^{K^{\text{I}}} \lambda_+(q, p_i) + \prod_{i=1}^{K^{\text{I}}} \lambda_-(q, p_i) \right\} + \\
& - \sum_{k=0}^{\ell_0-1} \prod_{m=1}^{K^{\text{II}}} \frac{\mathcal{X}_k^{k,0}(q, \lambda_m)}{\mathcal{Y}_{k;1}^{k,0;1}(q, \lambda_m)} \prod_{i=1}^{K^{\text{I}}} \mathcal{Z}_{k;1}^{k,0;1}(q, p_i) \Lambda^{(6v)}(u_0 + \frac{\ell_0-1}{2} - k, \vec{u}_\lambda), \quad (8.53)
\end{aligned}$$

where again $\Lambda^{(6v)}$ is the eigenvalue of the auxiliary 6-vertex model, and we have introduced $\vec{u}_\lambda = (u_{\lambda_1}, \dots, u_{\lambda_{K^{\text{II}}}})$. The auxiliary roots satisfy the following equations

$$\Lambda^{(6v)}(u_{\lambda_j}, \vec{u}_\lambda) \prod_{i=1}^{K^{\text{I}}} \mathcal{Z}_{0;1}^{0,0;1}(\lambda_j, p_i) = 1, \quad (8.54)$$

$$\prod_{i=1}^{K^{\text{II}}} b(w_j, u_{\lambda_i}) \prod_{i=1, i \neq j}^{K^{\text{III}}} \frac{b(w_i, w_j)}{b(w_j, w_i)} = 1. \quad (8.55)$$

In appendix B we give a complete derivation of the eigenvalues $\Lambda(q|\vec{p})$ and of the auxiliary equations for the case $K^{\text{III}} = 0$. Let us stress that the expression for $\Lambda(q|\vec{p})$ encodes many eigenvalues that are labelled by the integer quantum numbers. We would also like to mention that the form of the eigenvalues appears in the form of factorized products of single-excitation terms - a somewhat expected feature, which makes us more confident about the generalization procedure.

We point out that the dependence of the auxiliary parameters λ_m only appears in the form $x^+(\lambda_m)$. In order to compare with the known Bethe equations we relabel this to be $x^+(\lambda_m) \equiv y_m$.

We also rescale $w_i \rightarrow \frac{2i}{g}w_i$. In terms of these parameters, the eigenvalues (8.53) become

$$\begin{aligned} \Lambda(q|\vec{p}) = & \prod_{i=1}^{K^{\text{II}}} \frac{y_i - x_0^-}{y_i - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} + \\ & + \prod_{i=1}^{K^{\text{II}}} \frac{y_i - x_0^-}{y_i - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} \left[\frac{x_0^+ + \frac{1}{x_0^+} - y_i - \frac{1}{y_i}}{x_0^+ + \frac{1}{x_0^+} - y_i - \frac{1}{y_i} - \frac{2i\ell_0}{g}} \right] \prod_{i=1}^{K^{\text{I}}} \left[\frac{(x_0^- - x_i^-)(1 - x_0^- x_i^+)}{(x_0^- - x_i^+)(1 - x_0^+ x_i^-)} \sqrt{\frac{x_0^+ x_i^+}{x_0^- x_i^-}} \mathcal{X}_{\ell_0}^{\ell_0, 0} \right] \\ & + \sum_{k=1}^{\ell_0-1} \prod_{i=1}^{K^{\text{II}}} \frac{y_i - x_0^-}{y_i - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} \left[\frac{x_0^+ + \frac{1}{x_0^+} - y_i - \frac{1}{y_i}}{x_0^+ + \frac{1}{x_0^+} - y_i - \frac{1}{y_i} - \frac{2ik}{g}} \right] \left\{ \prod_{i=1}^{K^{\text{I}}} \lambda_+(q, p_i, k) + \prod_{i=1}^{K^{\text{I}}} \lambda_-(q, p_i, k) \right\} \\ & - \sum_{k=0}^{\ell_0-1} \prod_{i=1}^{K^{\text{II}}} \frac{y_i - x_0^-}{y_i - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} \left[\frac{x_0^+ + \frac{1}{x_0^+} - y_i - \frac{1}{y_i}}{x_0^+ + \frac{1}{x_0^+} - y_i - \frac{1}{y_i} - \frac{2ik}{g}} \right] \prod_{i=1}^{K^{\text{I}}} \frac{x_0^+ - x_i^+}{x_0^- - x_i^-} \sqrt{\frac{x_0^-}{x_0^+}} \left[1 - \frac{k}{u_0 - u_i + \frac{\ell_0 - \ell_i}{2}} \right] \times \\ & \times \mathcal{X}_k^{k, 0} \left\{ \prod_{i=1}^{K^{\text{III}}} \frac{w_i - x_0^+ - \frac{1}{x_0^+} + \frac{i(2k-1)}{g}}{w_i - x_0^+ - \frac{1}{x_0^+} + \frac{i(2k+1)}{g}} + \prod_{i=1}^{K^{\text{II}}} \frac{y_i + \frac{1}{y_i} - x_0^+ - \frac{1}{x_0^+} + \frac{2ik}{g}}{y_i + \frac{1}{y_i} - x_0^+ - \frac{1}{x_0^+} + \frac{2i(k+1)}{g}} \prod_{i=1}^{K^{\text{III}}} \frac{w_i - x_0^+ - \frac{1}{x_0^+} + \frac{i(2k+3)}{g}}{w_i - x_0^+ - \frac{1}{x_0^+} + \frac{i(2k+1)}{g}} \right\}. \end{aligned} \quad (8.56)$$

and the above auxiliary Bethe equations transform into the well-known ones (7.94):

$$\begin{aligned} \prod_{i=1}^{K^{\text{I}}} \frac{y_k - x_i^-}{y_k - x_i^+} \sqrt{\frac{x_i^+}{x_i^-}} &= \prod_{i=1}^{K^{\text{III}}} \frac{w_i - y_k - \frac{1}{y_k} - \frac{i}{g}}{w_i - y_k - \frac{1}{y_k} + \frac{i}{g}}, \\ \prod_{i=1}^{K^{\text{II}}} \frac{w_k - y_i - \frac{1}{y_i} + \frac{i}{g}}{w_k - y_i - \frac{1}{y_i} - \frac{i}{g}} &= \prod_{i=1, i \neq k}^{K^{\text{III}}} \frac{w_k - w_i + \frac{2i}{g}}{w_k - w_i - \frac{2i}{g}}. \end{aligned} \quad (8.57)$$

Once again, we find that for *all* particles in the fundamental representation (including the auxiliary space) this agrees with what obtained in [48]. Analogous to formula (41) from the same paper, one can derive the complete set of Bethe equations from the transfer matrix. One finds that the one-particle momenta should satisfy

$$e^{ip_j L} = \Lambda(p_j|\vec{p}). \quad (8.58)$$

One then notices that, if $q = p_j$ and $\ell_0 = \ell_j$, then $\mathcal{X}_k^{k, 0} = 0$ if $k > 0$. This means that the only surviving terms is found to be the first one. This gives the following Bethe equations (after explicitly including the appropriate scalar factor S_0 , which was omitted in the derivation):

$$e^{ip_j L} = \prod_{i=1, i \neq j}^{K^{\text{I}}} S_0(p_j, p_i) \prod_{m=1}^{K^{\text{II}}} \frac{y_m - x_j^-}{y_m - x_j^+} \sqrt{\frac{x_j^+}{x_j^-}}. \quad (8.59)$$

Together with the above set of auxiliary Bethe equations, this indeed reproduces the full set of Bethe equations (7.94).

8.3 Different vacua and fusion

In the previous sections we deduced the spectrum of the transfer matrix. We found all of its eigenstates and eigenvalues, characterized by the quantum numbers $K^{\text{I}, \text{II}, \text{III}}$. The eigenstates

were obtained by starting with a vacuum with quantum numbers $K^{\text{II}} = K^{\text{III}} = 0$, which proved to be an eigenstate, and then applying creation operators that generate eigenstates with different quantum numbers. Of course, our choice of vacuum is not unique. We can build up our algebraic Bethe ansatz starting from a different vacuum. One trivial example of this would be to start with w_2 instead of w_1 . A more interesting case arises when all physical particles are fermions.

8.3.1 Fermionic vacuum

Consider a fermionic vacuum with all the physical particles in the fundamental representation:

$$|0\rangle'_P = \theta_3 \otimes \dots \otimes \theta_3. \quad (8.60)$$

This vacuum has quantum numbers $K^{\text{II}} = K^{\text{I}}$ and $K^{\text{III}} = 0$. One can easily check that this vacuum is also an eigenstate. The action of the diagonal elements of fermionic type of the transfer matrix (8.9) is given by:

$$\begin{aligned} \mathcal{T}_{3;k}^{3;k} |0\rangle'_P &= \prod_{i=1}^{K^{\text{I}}} \mathcal{Z}_k^{k,0}(q, p_i) |0\rangle'_P, \\ \mathcal{T}_{4;k}^{4;k} |0\rangle'_P &= \prod_{i=1}^{K^{\text{I}}} \mathcal{Z}_{k;6}^{k,0;6}(q, p_i) |0\rangle'_P. \end{aligned}$$

The explicit values for these scattering elements is given in section 4.8, and one obtains

$$\begin{aligned} \mathcal{T}_{3;k}^{3;k} |0\rangle'_P &= \prod_{i=1}^{K^{\text{I}}} \frac{x_0^- - x_i^+}{x_0^+ - x_i^-} \sqrt{\frac{x_0^+ x_i^-}{x_0^- x_i^+}} |0\rangle'_P, \\ \mathcal{T}_{4;k}^{4;k} |0\rangle'_P &= \prod_{i=1}^{K^{\text{I}}} \frac{x_0^- - x_i^-}{x_0^+ - x_i^-} \frac{x_i^- - \frac{1}{x_0^+}}{x_i^+ - \frac{1}{x_0^+}} \sqrt{\frac{x_0^+ x_i^+}{x_0^- x_i^-}} |0\rangle'_P. \end{aligned} \quad (8.61)$$

Notice that these elements are *independent* of k . This means that, when summing over k , this will only give a factor of ℓ_0 .

The next step is to consider the bosonic elements $\mathcal{T}_k^k, \mathcal{T}_{34,k}^{34,k}$. Let us again split off the contributions from $k = 0$ and $k = \ell_0$. The corresponding elements $\mathcal{T}_0^0, \mathcal{T}_{\ell_0}^{\ell_0}$ act on this new vacuum as

$$\begin{aligned} \mathcal{T}_0^0 |0\rangle'_P = \mathcal{T}_{\ell_0}^{\ell_0} |0\rangle'_P &= \prod_{i=1}^{K^{\text{I}}} \mathcal{Z}_{k;2}^{k,0;2} |0\rangle'_P, \\ &= \prod_{i=1}^{K^{\text{I}}} \frac{x_0^- - x_i^-}{x_0^+ - x_i^-} \sqrt{\frac{x_0^+}{x_0^-}} |0\rangle'_P. \end{aligned} \quad (8.62)$$

For the remaining elements one finds again, as in the case of the vacuum (8.11) we have been using in the previous section, an additional matrix structure. More precisely, this time one needs

to compute the eigenvalues of the matrix

$$\begin{pmatrix} \mathcal{Y}_{k;2}^{k,0;2} & \mathcal{Y}_{k;2}^{k,0;4} \\ \mathcal{Y}_{k;4}^{k,0;2} & \mathcal{Y}_{k;4}^{k,0;4} \end{pmatrix}. \quad (8.63)$$

Because $\mathcal{Y}_{k;2}^{k,0;4} = \mathcal{Y}_{k;4}^{k,0;2} = 0$, one remarkably finds that this matrix is diagonal. Hence, the eigenvalues are easily read off, and one finds

$$\mathcal{T}_k^k |0\rangle'_P = \prod_{i=1}^{K^I} \mathcal{Y}_{k;2}^{k,0;2}(q, p_i) |0\rangle'_P = \prod_{i=1}^{K^I} \frac{x_0^- - x_i^-}{x_0^+ - x_i^-} \sqrt{\frac{x_0^+}{x_0^-}} |0\rangle'_P \quad (8.64)$$

and

$$\mathcal{T}_{34,k}^{34,k} |0\rangle'_P = \prod_{i=1}^{K^I} \mathcal{Y}_{k;4}^{k,0;4}(q, p_i) |0\rangle'_P = \prod_{i=1}^{K^I} \frac{x_0^- - x_i^+}{x_0^+ - x_i^-} \frac{x_i^- - \frac{1}{x_0^+}}{x_i^+ - \frac{1}{x_0^-}} \sqrt{\frac{x_0^+}{x_0^-}} |0\rangle'_P. \quad (8.65)$$

Similarly to the fermionic contributions (8.61), and once again in contrast to the bosonic vacuum, where we find a very non-trivial k -dependence through λ_{\pm} (8.30), one finds that these terms are *independent* of k . Summing everything up finally gives that $|0\rangle'_P$ is an eigenvalue of the transfer matrix with eigenvalue

$$\begin{aligned} \Lambda(q|\vec{p}) = & (\ell_0 + 1) \prod_{i=1}^{K^I} \frac{x_0^- - x_i^-}{x_0^+ - x_i^-} \sqrt{\frac{x_0^+}{x_0^-}} - \ell_0 \prod_{i=1}^{K^I} \frac{x_0^- - x_i^+}{x_0^+ - x_i^-} \sqrt{\frac{x_0^+ x_i^-}{x_0^- x_i^+}} - \\ & - \ell_0 \prod_{i=1}^{K^I} \frac{x_0^- - x_i^-}{x_0^+ - x_i^-} \frac{x_i^- - \frac{1}{x_0^+}}{x_i^+ - \frac{1}{x_0^-}} \sqrt{\frac{x_0^+ x_i^+}{x_0^- x_i^-}} + (\ell_0 - 1) \prod_{i=1}^{K^I} \frac{x_0^- - x_i^+}{x_0^+ - x_i^-} \frac{x_i^- - \frac{1}{x_0^+}}{x_i^+ - \frac{1}{x_0^-}} \sqrt{\frac{x_0^+}{x_0^-}}. \end{aligned} \quad (8.66)$$

This precisely agrees with the result of [55] for antisymmetric representations.

Let us remark that the spectrum is clearly independent of the choice of vacuum. Hence, one should find the same eigenvalues when starting from the vacuum $|0\rangle_P$ or from the vacuum $|0\rangle_{P'}$, provided one excites the appropriate set of auxiliary roots. In particular, if we were to reproduce (8.66) starting from the bosonic vacuum and exciting enough fermions, we would have to first solve the K^{II} auxiliary BAE, and then use these solutions to find the corresponding eigenvalue, which should therefore agree with (8.66). In fact, conversion of one eigenvalue into the other can be obtained by means of duality transformations [152]. We would also like to notice that the result obtained in this section for fundamental representations in the physical space happens to have nice fusion properties, and one can think of combining several of such elementary transfer matrices to obtain more general ones. This approach has been followed for instance in [153].

8.3.2 Bosonic vacuum

Let us now come back to the bosonic vacuum (8.11) we have been using in the first part of this chapter. In [55], a prescription for computing the transfer matrix eigenvalues, for all physical

particles in the fundamental representation, was also given. The formula was expressed in terms of an expansion of the inverse of a quantum characteristic function. We have found that this prescription indeed produces the same eigenvalues as obtained from our general formula (8.56), when restricting the latter to fundamental particles in the physical space. To demonstrate this fact, we explicitly work out here below the above mentioned expansion following [55], adapting the calculation to the notations we use here. We will then compare the final formula with the suitable restriction of our result (8.56), finding perfect agreement. Indeed, we will be able to relax the condition of physical legs in the fundamental representation, by making the conjectured expression for the quantum characteristic function slightly more general. We will then find agreement with such a formula in the general case where we are dealing with generic bound state representations $\ell_i \neq 1$ as well.

Following [55], we define the shift operator U by

$$U f(u) U^{-1} = f\left(u + \frac{1}{2}\right), \quad (8.67)$$

and introduce the notation

$$f^{[\ell]}(u) \equiv U^\ell f(u) U^{-\ell} = f\left(u + \frac{\ell}{2}\right). \quad (8.68)$$

The spectral parameters of an elementary particle, defined in (8.16), satisfy the relation

$$x^{[1]} + \frac{1}{x^{[1]}} - x^{[-1]} - \frac{1}{x^{[-1]}} = \frac{2i}{g}. \quad (8.69)$$

By successive applications of the shift operator to (8.69), one finds that the pair of variables $\{x^{[\ell]}, x^{[\ell-2k]}\}$ defines another rapidity torus

$$x^{[\ell]} + \frac{1}{x^{[\ell]}} - x^{[\ell-2k]} - \frac{1}{x^{[\ell-2k]}} = \frac{2ik}{g}. \quad (8.70)$$

There are two choices of branch for $x_a^{[\ell-2k]}$ for a given $x^{[\ell]}$, as can be seen by

$$x^{[\ell-2k]} = \frac{1}{2} \left(x^{[\ell]} + \frac{1}{x^{[\ell]}} - \frac{2ik}{g} + \sqrt{\left(x^{[\ell]} + \frac{1}{x^{[\ell]}} - \frac{2ik}{g} \right)^2 - 4} \right). \quad (8.71)$$

We also use $y_i + 1/y_i = iv_i$ in what follows.⁵

Let $\langle \ell_0 - 1, 0 \rangle$ be the ℓ_0 -th symmetric representation of $\mathfrak{su}(2|2)$. The conjecture states that the transfer matrix for such a representation $T_{\langle \ell_0 - 1, 0 \rangle}(u_0 | \{\vec{u}, \vec{v}, \vec{w}\})$ is generated by $T_{\langle 0, 0 \rangle}(u_0 | \{\vec{u}, \vec{v}, \vec{w}\})$,

⁵ Interestingly, the final result (8.89) is almost invariant under the map $y_i \mapsto 1/y_i$, except for an overall factor.

where the generating function is equal to the inverse of the quantum characteristic function:

$$D_0^{-1} := (1 - U_0 T_4 U_0)^{-1} (1 - U_0 T_3 U_0) (1 - U_0 T_2 U_0) (1 - U_0 T_1 U_0)^{-1}, \quad (8.72)$$

$$\begin{aligned} &= \left(1 + \sum_{h=1}^{\infty} (U_0 T_4 U_0)^h\right) (1 - U_0 T_3 U_0) (1 - U_0 T_2 U_0) \left(1 + \sum_{k=1}^{\infty} (U_0 T_1 U_0)^k\right), \\ &\equiv \sum_{\ell_0=0}^{\infty} U_0^{\ell_0} T_{\langle \ell_0-1, 0 \rangle} (u_0 | \{\vec{u}, \vec{v}, \vec{w}\}) U_0^{\ell_0}. \end{aligned} \quad (8.73)$$

where the T_i s are parts of the fundamental transfer matrix, which we will specify later. Here U_0 is the shift operator for the variable u_0 . The first few terms can be found as follows:

$$\begin{aligned} D_0^{-1} &= 1 + U_0 (T_4 - T_3 - T_2 + T_1) U_0 \\ &+ U_0^2 \left\{ T_4^{[-1]} T_4^{[1]} + T_4^{[-1]} T_1^{[1]} + T_1^{[-1]} T_1^{[1]} + T_3^{[-1]} T_2^{[1]} \right. \\ &\quad \left. - T_4^{[-1]} (T_3^{[1]} + T_2^{[1]}) - (T_3^{[-1]} + T_2^{[-1]}) T_1^{[1]} \right\} U_0^2 + \dots, \end{aligned} \quad (8.74)$$

and, in general,

$$T_{\langle \ell_0-1, 0 \rangle} (u_0 | \{\vec{u}, \vec{v}, \vec{w}\}) = \tau_{\ell_0, 0} - \tau_{\ell_0, 1} [T_3] - \tau_{\ell_0, 1} [T_2] + \tau_{\ell_0, 2} [T_3, T_2], \quad (8.75)$$

where

$$\tau_{\ell_0, 0} = \sum_{k=0}^{\ell_0} T_4^{[-\ell_0+1]} T_4^{[-\ell_0+3]} \dots T_4^{[\ell_0-2k-3]} T_1^{[\ell_0-2k-1]} \dots T_1^{[\ell_0-1]}, \quad (8.76)$$

$$\tau_{\ell_0, 1} [X] = \sum_{k=0}^{\ell_0-1} T_4^{[-\ell_0+1]} T_4^{[-\ell_0+3]} \dots T_4^{[-\ell_0+2k-1]} X^{[\ell_0-2k-1]} T_1^{[\ell_0-2k+1]} \dots T_1^{[\ell_0-1]}, \quad (8.77)$$

$$\tau_{\ell_0, 2} [X, Y] = \sum_{k=0}^{\ell_0-2} T_4^{[-\ell_0+1]} T_4^{[-\ell_0+3]} \dots T_4^{[-\ell_0+2k-1]} X^{[\ell_0-2k-3]} Y^{[\ell_0-2k-1]} T_1^{[\ell_0-2k+1]} \dots T_1^{[\ell_0-1]}. \quad (8.78)$$

The first line of (8.74) gives the transfer matrix for the fundamental representation as

$$T_{\langle 0, 0 \rangle} (u_0 | \{\vec{u}, \vec{v}, \vec{w}\}) = T_1 - T_2 - T_3 + T_4. \quad (8.79)$$

We recall that the left hand side of this equation is given explicitly by (8.56) at $\ell_0 = 1$, which

reads

$$\begin{aligned}
\Lambda(q|\vec{p}) = & \prod_{i=1}^{K^{\text{II}}} \frac{y_i - x_0^-}{y_i - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} + \\
& + \prod_{i=1}^{K^{\text{II}}} \frac{y_i - x_0^-}{y_i - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} \left[\frac{x_0^+ + \frac{1}{x_0^+} - y_i - \frac{1}{y_i}}{x_0^+ + \frac{1}{x_0^+} - y_i - \frac{1}{y_i} - \frac{2i}{g}} \right] \prod_{i=1}^{K^{\text{I}}} \left[\frac{(x_0^- - x_i^-)(1 - x_0^- x_i^+)}{(x_0^- - x_i^+)(1 - x_0^+ x_i^-)} \sqrt{\frac{x_0^+ x_i^+}{x_0^- x_i^-}} \mathcal{X}_1^{1,0} \right] \\
& - \prod_{i=1}^{K^{\text{II}}} \frac{y_i - x_0^-}{y_i - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} \prod_{i=1}^{K^{\text{I}}} \frac{x_0^+ - x_i^+}{x_0^+ - x_i^-} \sqrt{\frac{x_0^-}{x_0^+}} \times \\
& \times \mathcal{X}_0^{0,0} \left\{ \prod_{i=1}^{K^{\text{III}}} \frac{w_i - x_0^+ - \frac{1}{x_0^+} - \frac{i}{g}}{w_i - x_0^+ - \frac{1}{x_0^+} + \frac{i}{g}} + \prod_{i=1}^{K^{\text{II}}} \frac{y_i + \frac{1}{y_i} - x_0^+ - \frac{1}{x_0^+}}{y_i + \frac{1}{y_i} - x_0^+ - \frac{1}{x_0^+} + \frac{2i}{g}} \prod_{i=1}^{K^{\text{III}}} \frac{w_i - x_0^+ - \frac{1}{x_0^+} + \frac{3i}{g}}{w_i - x_0^+ - \frac{1}{x_0^+} + \frac{i}{g}} \right\}.
\end{aligned} \tag{8.80}$$

Therefore, $\Lambda(q|\vec{p})$ may be equated with the right hand side of (8.79) term by term. We simplify the above expression of $\Lambda(q|\vec{p})$ by introducing variables w_i and v_i as follows⁶:

$$w_i - x_0^\pm - \frac{1}{x_0^\mp} + \frac{in}{g} \equiv \left(w_i - u_0 + \frac{\mp \ell_0 + n}{2} \right) \frac{2i}{g}, \tag{8.81}$$

$$y_i + \frac{1}{y_i} - x_0^\pm - \frac{1}{x_0^\mp} + \frac{in}{g} \equiv \left(v_i - u_0 + \frac{\mp \ell_0 + n}{2} \right) \frac{2i}{g}. \tag{8.82}$$

With the help of (8.15) it produces

$$\begin{aligned}
\Lambda(q|\vec{p}) = & \prod_{i=1}^{K^{\text{II}}} \frac{y_i - x_0^-}{y_i - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} \\
& + \prod_{i=1}^{K^{\text{II}}} \frac{y_i - x_0^-}{y_i - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} \left[\frac{v_i - u_0 - \frac{1}{2}}{v_i - u_0 + \frac{1}{2}} \right] \prod_{i=1}^{K^{\text{I}}} \frac{(x_0^+ - x_i^+) \left(1 - \frac{1}{x_0^- x_i^+} \right)}{(x_0^+ - x_i^-) \left(1 - \frac{1}{x_0^- x_i^-} \right)} \\
& - \prod_{i=1}^{K^{\text{II}}} \frac{y_i - x_0^-}{y_i - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} \prod_{i=1}^{K^{\text{I}}} \frac{x_0^+ - x_i^+}{x_0^+ - x_i^-} \sqrt{\frac{x_i^-}{x_i^+}} \times \\
& \times \left\{ \prod_{i=1}^{K^{\text{III}}} \frac{w_i - u_0 - 1}{w_i - u_0} + \prod_{i=1}^{K^{\text{II}}} \frac{v_i - u_0 - \frac{1}{2}}{v_i - u_0 + \frac{1}{2}} \prod_{i=1}^{K^{\text{III}}} \frac{w_i - u_0 + 1}{w_i - u_0} \right\}.
\end{aligned} \tag{8.83}$$

It is useful to separate a common factor in the following fashion:

$$T_i = S_{\langle 0,0 \rangle} \tilde{T}_i, \quad S_{\langle 0,0 \rangle} \equiv \prod_{i=1}^{K^{\text{II}}} \frac{y_i - x_0^-}{y_i - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}}, \quad (i = 1, \dots, 4). \tag{8.84}$$

⁶Our notation is $x_0^\pm = x_0^{[\pm \ell_0]}$, and $\ell_0 = 1$ is used when discussing the fundamental transfer matrix. Note that the shift operator does not act on x_i^\pm .

Then, the tilded functions can be written as

$$\tilde{T}_1 = \prod_{i=1}^{K^{\text{II}}} \frac{v_i - u_0 - \frac{1}{2}}{v_i - u_0 + \frac{1}{2}} \prod_{i=1}^{K^{\text{I}}} \frac{\left(1 - \frac{1}{x_0^- x_i^+}\right) (x_0^+ - x_i^+)}{\left(1 - \frac{1}{x_0^- x_i^-}\right) (x_0^+ - x_i^-)}, \quad (8.85)$$

$$\tilde{T}_2 = \prod_{i=1}^{K^{\text{III}}} \frac{w_i - u_0 + 1}{w_i - u_0} \prod_{i=1}^{K^{\text{II}}} \frac{v_i - u_0 - \frac{1}{2}}{v_i - u_0 + \frac{1}{2}} \prod_{i=1}^{K^{\text{I}}} \frac{x_0^+ - x_i^+}{x_0^+ - x_i^-} \sqrt{\frac{x_i^-}{x_i^+}}, \quad (8.86)$$

$$\tilde{T}_3 = \prod_{i=1}^{K^{\text{III}}} \frac{w_i - u_0 - 1}{w_i - u_0} \prod_{i=1}^{K^{\text{I}}} \frac{x_0^+ - x_i^+}{x_0^+ - x_i^-} \sqrt{\frac{x_i^-}{x_i^+}}, \quad (8.87)$$

$$\tilde{T}_4 = 1. \quad (8.88)$$

Note that different identification of \tilde{T}_i 's would produce the transfer matrix for different representations [152].

One can now explicitly evaluate the the function τ 's appearing in the conjectured transfer matrix for the ℓ_0 -th symmetric representation (8.75). We will only state the final result here and refer for details to [154]. The transfer matrix derived from the conjecture on the quantum characteristic function:

$$T_{\langle \ell_0 - 1, 0 \rangle}(u_0 | \{\vec{u}, \vec{v}, \vec{w}\}) = \prod_{i=1}^{K^{\text{II}}} \frac{y_i - x_0^{[-\ell_0]}}{y_i - x_0^{[\ell_0]}} \sqrt{\frac{x_0^{[\ell_0]}}{x_0^{[-\ell_0]}}} \times \quad (8.89)$$

$$\begin{aligned} & \left(1 + \prod_{i=1}^{K^{\text{II}}} \frac{v_i - u_0 - \frac{\ell_0}{2}}{v_i - u_0 + \frac{\ell_0}{2}} \prod_{i=1}^{K^{\text{I}}} \left[\frac{x_0^{[-\ell_0]} - x_i^-}{x_0^{[\ell_0]} - x_i^-} \frac{1 - \frac{1}{x_0^{[-\ell_0]} x_i^+}}{1 - \frac{1}{x_0^{[\ell_0]} x_i^+}} \frac{\mathcal{X}_{\ell_0}^{\ell_0, 0}}{\mathcal{D}} \right] \right. \\ & + \sum_{k=1}^{\ell_0 - 1} \left\{ \prod_{i=1}^{K^{\text{II}}} \frac{v_i - u_0 - \frac{\ell_0}{2}}{v_i - u_0 - \frac{\ell_0 - 2k}{2}} \prod_{i=1}^{K^{\text{I}}} \left[\frac{x_0^{[\ell_0 - 2k]} - x_i^-}{x_0^{[\ell_0 - 2k]} - x_i^+} \frac{u_0 - u_i + \frac{\ell_0 - \ell_i - 2k}{2}}{u_0 - u_i + \frac{\ell_0 - \ell_i}{2}} \frac{x_0^{[\ell_0]} - x_i^+}{x_0^{[\ell_0]} - x_i^-} \frac{\mathcal{X}_k^{k, 0}}{\mathcal{D}} \right] \right\} \\ & + \sum_{k=1}^{\ell_0 - 1} \left\{ \prod_{i=1}^{K^{\text{II}}} \frac{v_i - u_0 - \frac{\ell_0}{2}}{v_i - u_0 - \frac{\ell_0 - 2k}{2}} \prod_{i=1}^{K^{\text{I}}} \left[\frac{x_i^-}{x_i^+} \frac{1 - \frac{1}{x_0^{[\ell_0 - 2k]} x_i^-}}{1 - \frac{1}{x_0^{[\ell_0 - 2k]} x_i^+}} \frac{u_0 - u_i + \frac{\ell_0 - \ell_i - 2k}{2}}{u_0 - u_i + \frac{\ell_0 - \ell_i}{2}} \frac{x_0^{[\ell_0]} - x_i^+}{x_0^{[\ell_0]} - x_i^-} \frac{\mathcal{X}_k^{k, 0}}{\mathcal{D}} \right] \right\} \\ & - \sum_{k=0}^{\ell_0 - 1} \left\{ \prod_{i=1}^{K^{\text{III}}} \frac{w_i - u_0 - \frac{\ell_0 - 2k + 1}{2}}{w_i - u_0 - \frac{\ell_0 - 2k - 1}{2}} \prod_{i=1}^{K^{\text{II}}} \frac{v_i - u_0 - \frac{\ell_0}{2}}{v_i - u_0 - \frac{\ell_0 - 2k}{2}} \prod_{i=1}^{K^{\text{I}}} \left[\sqrt{\frac{x_i^-}{x_i^+}} \frac{u_0 - u_i + \frac{\ell_0 - \ell_i - 2k}{2}}{u_0 - u_i + \frac{\ell_0 - \ell_i}{2}} \frac{x_0^{[\ell_0]} - x_i^+}{x_0^{[\ell_0]} - x_i^-} \frac{\mathcal{X}_k^{k, 0}}{\mathcal{D}} \right] \right\} \\ & - \sum_{k=0}^{\ell_0 - 1} \left\{ \prod_{i=1}^{K^{\text{III}}} \frac{w_i - u_0 - \frac{\ell_0 - 2k - 3}{2}}{w_i - u_0 - \frac{\ell_0 - 2k - 1}{2}} \prod_{i=1}^{K^{\text{II}}} \frac{v_i - u_0 - \frac{\ell_0}{2}}{v_i - u_0 - \frac{\ell_0 - 2k - 2}{2}} \prod_{i=1}^{K^{\text{I}}} \left[\sqrt{\frac{x_i^-}{x_i^+}} \frac{u_0 - u_i + \frac{\ell_0 - \ell_i - 2k}{2}}{u_0 - u_i + \frac{\ell_0 - \ell_i}{2}} \frac{x_0^{[\ell_0]} - x_i^+}{x_0^{[\ell_0]} - x_i^-} \frac{\mathcal{X}_k^{k, 0}}{\mathcal{D}} \right] \right\} \Bigg). \end{aligned}$$

Finally by substituting $x^{[\ell_0-2k]}$ of (8.71) into the definition of $\lambda_{\pm}(q, p_i, k)$ in (8.30), we find

$$\lambda_{\pm}(q, p_i, k) = \begin{cases} \frac{x_0^{[\ell_0-2k]} - x_i^-}{x_0^{[\ell_0-2k]} - x_i^+} \frac{u_0 - u_i + \frac{\ell_0 - \ell_i - 2k}{2}}{u_0 - u_i + \frac{\ell_0 - \ell_i}{2}} \frac{x_0^{[\ell_0]} - x_i^+}{x_0^{[\ell_0]} - x_i^-} \frac{\mathcal{X}_k^{k,0}}{\mathcal{D}}, \\ \frac{x_i^-}{x_i^+} \frac{1 - \frac{1}{x_0^{[\ell_0-2k]} x_i^-}}{1 - \frac{1}{x_0^{[\ell_0-2k]} x_i^+}} \frac{u_0 - u_i + \frac{\ell_0 - \ell_i - 2k}{2}}{u_0 - u_i + \frac{\ell_0 - \ell_i}{2}} \frac{x_0^{[\ell_0]} - x_i^+}{x_0^{[\ell_0]} - x_i^-} \frac{\mathcal{X}_k^{k,0}}{\mathcal{D}}. \end{cases} \quad (8.90)$$

From this one can compare the above result term by term with the previously derived result and find agreement.

How the agreement works can be understood in the following way. From the expression (8.89) we see that, apparently, a spurious dependence on the parameters $x_0^{[\ell_0-2k]}$ is left among the different blocks of the quantum characteristic function. However, one can make use of (8.71) to re-express each of these variables only in terms of the bound state variable $x_0^{[\ell_0]}$, provided one chooses a branch of the quadratic map. The remarkable observation is that, after this replacement, one can recast the above expression in a form that precisely agrees with our result (8.56). This happens for both choices of branch, consistent with the fact that the formula we have obtained *via* the alternative route of the ABA does not bear any dependence on such a choice.

Acknowledgements

First and foremost I would like to thank G. Arutyunov for many valuable discussions and for sharing insights on this subject. I am grateful to A. Torrielli, S. Frolov and R. Suzuki for fruitful discussions and collaborations. I would also like to thank Z. Bajnok, N. Beisert, B. de Wit, S. Frolov and M. Staudacher for giving useful comments on the manuscript.

Appendix A: Algebraic Bethe ansatz for the 6-vertex model

In this chapter we used the algebraic Bethe ansatz approach to diagonalize the $\text{AdS}_5 \times \text{S}^5$ superstring transfer matrix for bound states. We closely followed the discussion for the Hubbard model [150]. In this model, just as in our case, the 6-vertex model plays an important role. In this section we will discuss the algebraic Bethe ansatz for this model, for completeness and to fix notations.

The algebraic Bethe ansatz for the 6-vertex model is a standard chapter of the theory of integrable systems, and it is treated for example in [50, 120]. The scattering matrix of the model

is given by

$$r_{12}(u_1, u_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u_1, u_2) & a(u_1, u_2) & 0 \\ 0 & a(u_1, u_2) & b(u_1, u_2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (8.91)$$

where

$$a = \frac{U}{u_1 - u_2 + U}, \quad b = \frac{u_1 - u_2}{u_1 - u_2 + U}. \quad (8.92)$$

It is convenient to write it as

$$\begin{aligned} r_{12}(u_1, u_2) &= r_{\alpha\beta}^{\gamma\delta}(u_1, u_2) E_\gamma^\alpha \otimes E_\delta^\beta \\ &= \frac{u_1 - u_2}{u_1 - u_2 + U} \left[E_\alpha^\alpha \otimes E_\beta^\beta + \frac{U}{u_1 - u_2} E_\beta^\alpha \otimes E_\alpha^\beta \right], \end{aligned} \quad (8.93)$$

with E_β^α the standard matrix unities (the matrices where all entries are zero except for a 1 at position (β, α)). Let us consider K particles, with rapidities u_i . One can construct the monodromy matrix

$$\mathcal{T}(u_0|\vec{u}) = \prod_{i=1}^K r_{0i}(u_0|u_i). \quad (8.94)$$

Let us write it as a matrix in the auxiliary space

$$\mathcal{T}^{(1)}(u_0|\vec{u}) = \begin{pmatrix} A(u_0|\vec{u}) & B(u_0|\vec{u}) \\ C(u_0|\vec{u}) & D(u_0|\vec{u}) \end{pmatrix}. \quad (8.95)$$

In the algebraic Bethe Ansatz, one constructs the eigenvalues of the transfer matrix by first specifying a ground state $|0\rangle$. The ground state, in this case, is defined as

$$|0\rangle = \bigotimes_{i=1}^K \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (8.96)$$

It is easily checked that it is an eigenstate of the transfer matrix. More precisely, the action of the different elements of the monodromy matrix on $|0\rangle$ is given by

$$\begin{aligned} A(u_0|\vec{u})|0\rangle &= |0\rangle, \\ C(u_0|\vec{u})|0\rangle &= 0, \\ D(u_0|\vec{u})|0\rangle &= \prod_{i=1}^K b(u_0, u_i)|0\rangle. \end{aligned} \quad (8.97)$$

Thus, $|0\rangle$ is an eigenstate of the transfer matrix with the following eigenvalue

$$1 + \prod_{i=1}^K b(u_0, u_i). \quad (8.98)$$

The operator B from the monodromy matrix will be considered as a creation operator. It will create all the other eigenstates out of the vacuum. We introduce additional parameters w_i and consider the state

$$|M\rangle := \phi_M(w_1, \dots, w_M)|0\rangle, \quad \phi_M(w_1, \dots, w_M) := \prod_{i=1}^M B(w_i|\vec{u}). \quad (8.99)$$

In the context of the Heisenberg spin chain the vacuum corresponds to all spins down and the state $|M\rangle$ corresponds to the eigenstate of the transfer matrix that has M spins turned up.

In order to evaluate the action of the transfer matrix $\mathcal{T}(u_0|\vec{u}) = A(u_0|\vec{u}) + D(u_0|\vec{u})$ on the state $|M\rangle$, one needs the commutation relations between the fields A, B, D . From (8.6) applied to this S-matrix (8.91) one reads

$$\begin{aligned} A(u_0|\vec{u})B(w|\vec{u}) &= \frac{1}{b(w, u_0)}B(w|\vec{u})A(u_0|\vec{u}) - \frac{a(w, u_0)}{b(w, u_0)}B(u_0|\vec{u})A(w|\vec{u}) \\ B(w_1|\vec{u})B(w_2|\vec{u}) &= B(w_2|\vec{u})B(w_1|\vec{u}) \\ D(u_0|\vec{u})B(w|\vec{u}) &= \frac{1}{b(u_0, w)}B(w|\vec{u})D(u_0|\vec{u}) - \frac{a(u_0, w)}{b(u_0, w)}B(u_0|\vec{u})D(w|\vec{u}). \end{aligned} \quad (8.100)$$

From this, one can determine exactly when $|M\rangle$ is an eigenstate of the transfer matrix. By definition we have that

$$|M\rangle = B(w_M|\vec{u})|M-1\rangle, \quad (8.101)$$

and this allows us to use induction. By using the identity

$$\frac{1}{b(w_M, u_0)} \frac{a(w_i, u_0)}{b(w_i, u_0)} - \frac{a(w_M, u_0)}{b(w_M, u_0)} \frac{a(w_i, w_M)}{b(w_i, w_M)} = \frac{a(w_i, u_0)}{b(w_i, u_0)} \frac{1}{b(w_M, w_i)} \quad (8.102)$$

in (8.101) one can prove

$$\begin{aligned} A(u_0|\vec{u})\phi_M(w_1, \dots, w_M) &= \prod_{i=1}^M \frac{1}{b(w_i, u_0)} \phi_M(w_1, \dots, w_M) A(u_0|\vec{u}) \\ &\quad - \sum_{i=1}^M \left[\frac{a(w_i, u_0)}{b(w_i, u_0)} \prod_{j=1, j \neq i}^M \frac{1}{b(w_j, w_i)} \hat{\phi}_M A(w_i|\vec{u}) \right], \end{aligned} \quad (8.103)$$

where $\hat{\phi}_M$ stands for $\phi_M(\dots, w_{i-1}, u_0, w_{i+1}, \dots)$. One can find a similar relation for the com-

mutator between D and B . Using these relations gives

$$\begin{aligned} \mathcal{T}(u_0|\vec{u})|M\rangle &= \{A(u_0|\vec{u}) + D(u_0|\vec{u})\}|M\rangle \\ &= \phi_M(w_1, \dots, w_M) \left\{ A(u_0|\vec{u}) \prod_{i=1}^M \frac{1}{b(w_i, u_0)} + D(u_0|\vec{u}) \prod_{i=1}^M \frac{1}{b(u_0, w_i)} \right\} |0\rangle \\ &\quad - \sum_{i=1}^M \left[\frac{a(w_i, u_0)}{b(w_i, u_0)} \hat{\phi}_M \left\{ \prod_{j \neq i} \frac{1}{b(w_j, w_i)} A(w_i|\vec{u}) - \prod_{j \neq i} \frac{1}{b(w_i, w_j)} D(w_i|\vec{u}) \right\} \right] |0\rangle. \end{aligned} \quad (8.104)$$

From this we find that $|M\rangle$ is an eigenstate of the transfer matrix with eigenvalue

$$\Lambda^{(6v)}(u_0|\vec{u}) = \prod_{i=1}^M \frac{1}{b(w_i, u_0)} + \prod_{i=1}^M \frac{1}{b(u_0, w_i)} \prod_{i=1}^K b(u_0, u_i) \quad (8.105)$$

provided that the auxiliary parameters w_i satisfy the following equations

$$\prod_{i=1}^K b(w_j, u_i) = \prod_{i=1, i \neq j}^M \frac{b(w_j, w_i)}{b(w_i, w_j)}. \quad (8.106)$$

This now completely determines the spectrum of the 6-vertex model.

To conclude, we briefly explain how these eigenvalues are used to generate an infinite tower of conserved charges. From (8.6) one finds that

$$\mathcal{T}(u_0|\vec{u})\mathcal{T}(\mu|\vec{u}) = \mathcal{T}(\mu|\vec{u})\mathcal{T}(u_0|\vec{u}). \quad (8.107)$$

This means that if one writes $\mathcal{T}(u_0|\vec{u})$ as a series the auxiliary parameter u_0 , the coefficients of this series will depend on \vec{u} and they mutually commute. It actually turns out that the 6-vertex model Hamiltonian can be written in terms of these coefficients.

Appendix B: Excited states, $K^{\text{III}} = 0$

In this section we will discuss the class of higher excited states with $K^{\text{III}} = 0$. We will present for these states a full derivation of transfer matrix eigenvalues and auxiliary Bethe equations. From the general construction it is easily seen that a more general eigenvector of the transfer matrix is given by

$$|a\rangle = \Phi(\lambda_1, \dots, \lambda_a)|0\rangle_P, \quad \Phi(\lambda_1, \dots, \lambda_a) = B_3(\lambda_1) \dots B_3(\lambda_a). \quad (8.108)$$

These states have quantum number $K^{\text{III}} = 0$. This allows for a similar inductive procedure as applied to the 6-vertex model in appendix A. Furthermore, because of the properties of the creation operators (8.43), we find that

$$\Phi(\lambda_1, \dots, \lambda_{j-1}, \lambda_j, \dots, \lambda_a) = -\mathcal{X}_0^{0,0} r_{33}^{33}(\lambda_{j-1}, \lambda_j) \Phi(\lambda_1, \dots, \lambda_j, \lambda_{j-1}, \dots, \lambda_a). \quad (8.109)$$

This means that all permutations of the momenta λ_i are related to each other by a simple multiplication by a scalar prefactor. We will exploit this property later on. Let us first derive some useful identities. One uses induction to show that

$$A_3^4|a\rangle = C_3^*|a\rangle = C_4|a\rangle = C|a\rangle = 0, \quad (8.110)$$

for any a . This vastly simplifies the computations, since we can discard any term proportional to the above operators from the commutation relations. Let us first turn to (8.40). This now becomes, after discarding the term proportional to C_3^* ,

$$\mathcal{T}_{3,k}^{3,k}(q)B_3(\lambda) = \frac{\mathcal{X}_k^{k,0}}{\mathcal{Y}_{k;1}^{k,0;1}}B_3(\lambda)\mathcal{T}_{3,k}^{3,k}(q) + \frac{\mathcal{Y}_{k;2}^{k,1;1}}{\mathcal{Y}_{k;1}^{k,0;1}}\mathcal{T}_{3,k}^k(q)A_3^3(\lambda) + \frac{\mathcal{Y}_{k;4}^{k,1;1}}{\mathcal{Y}_{k;1}^{k,0;1}}\mathcal{T}_{3,k}^{34,k-1}(q)A_3^3(\lambda).$$

Applying this to $\Phi(\lambda_1, \dots, \lambda_a) = B_3(\lambda_1)\Phi(\lambda_2, \dots, \lambda_a)$ we find

$$\begin{aligned} \mathcal{T}_{3,k}^{3,k}(q)\Phi(\lambda_1, \dots, \lambda_a) &= \frac{\mathcal{X}_k^{k,0}}{\mathcal{Y}_{k;1}^{k,0;1}}B_3(\lambda_1)\mathcal{T}_{3,k}^{3,k}(q)\Phi(\lambda_2, \dots, \lambda_a) \\ &+ \frac{\mathcal{Y}_{k;2}^{k,1;1}}{\mathcal{Y}_{k;1}^{k,0;1}}\mathcal{T}_{3,k}^k(q)A_3^3(\lambda_1)\Phi(\lambda_2, \dots, \lambda_a) \\ &+ \frac{\mathcal{Y}_{k;4}^{k,1;1}}{\mathcal{Y}_{k;1}^{k,0;1}}\mathcal{T}_{3,k}^{34,k-1}(q)A_3^3(\lambda_1)\Phi(\lambda_2, \dots, \lambda_a). \end{aligned} \quad (8.111)$$

Obviously, by applying this relation recursively one finds

$$\begin{aligned} \mathcal{T}_{3,k}^{3,k}(q)\Phi(\lambda_1, \dots, \lambda_a) &= \prod_{i=1}^a \frac{\mathcal{X}_k^{k,0}(q, \lambda_i)}{\mathcal{Y}_{k;1}^{k,0;1}(q, \lambda_i)}\Phi(\lambda_1, \dots, \lambda_a)\mathcal{T}_{3,k}^{3,k}(q) \\ &+ \sum_{i=1}^a c_i \Phi_{k;i}(q, \lambda)A_3^3(\lambda_i) + \sum_{i=1}^a d_i \Psi_{k;i}(q, \lambda)A_3^3(\lambda_i), \end{aligned} \quad (8.112)$$

where c_i are some numerical coefficients and $\Phi_{k;i}(q, \lambda) = \mathcal{T}_{3,k}^k(q) \prod_{j \neq i} B_3(\lambda_j)$, $\Psi_{k;i}(q, \lambda) = \mathcal{T}_{3,k}^{34,k-1}(q) \prod_{j \neq i} B_3(\lambda_j)$. It is easily seen from (8.111) that the numerical coefficients in front of $\Phi_{k;1}(q, \lambda)$, $\Psi_{k;1}(q, \lambda)$ are given by

$$c_1 = \frac{\mathcal{Y}_{k;2}^{k,1;1}(q, \lambda_1)}{\mathcal{Y}_{k;1}^{k,0;1}(q, \lambda_1)} \prod_{i=2}^a \frac{\mathcal{X}_k^{k,0}(q, \lambda_i)}{\mathcal{Y}_{k;1}^{k,0;1}(q, \lambda_i)}, \quad d_1 = \frac{\mathcal{Y}_{k;4}^{k,1;1}(q, \lambda_1)}{\mathcal{Y}_{k;1}^{k,0;1}(q, \lambda_1)} \prod_{i=2}^a \frac{\mathcal{X}_k^{k,0}(q, \lambda_i)}{\mathcal{Y}_{k;1}^{k,0;1}(q, \lambda_i)}. \quad (8.113)$$

Here we can exploit the symmetry property (8.109) to relate all the other coefficients to this one. Let us denote these proportionality coefficients by \mathcal{P}_{1i} . We find

$$\begin{aligned} \mathcal{T}_{3,k}^{3,k}(q)\Phi(\lambda_1, \dots, \lambda_a) &= \prod_{i=1}^a \frac{\mathcal{X}_k^{k,0}(q, \lambda_i)}{\mathcal{Y}_{k;1}^{k,0;1}(q, \lambda_i)}\Phi(\lambda_1, \dots, \lambda_a)\mathcal{T}_{3,k}^{3,k}(q) \\ &+ \sum_{i=1}^a c_i \mathcal{P}_{1i} \Phi_{k;i}(q, \lambda)A_3^3(\lambda_i) + \sum_{i=1}^a d_i \mathcal{P}_{1i} \Psi_{k;i}(q, \lambda)A_3^3(\lambda_i), \end{aligned} \quad (8.114)$$

where

$$c_j = \frac{\mathcal{Y}_{k;2}^{k,1;1}(q, \lambda_j)}{\mathcal{Y}_{k;1}^{k,0;1}(q, \lambda_j)} \prod_{i=1, i \neq j}^a \frac{\mathcal{X}_k^{k,0}(q, \lambda_i)}{\mathcal{Y}_{k;1}^{k,0;1}(q, \lambda_i)}, \quad d_j = \frac{\mathcal{Y}_{k;4}^{k,1;1}(q, \lambda_j)}{\mathcal{Y}_{k;1}^{k,0;1}(q, \lambda_j)} \prod_{i=1, i \neq j}^a \frac{\mathcal{X}_k^{k,0}(q, \lambda_i)}{\mathcal{Y}_{k;1}^{k,0;1}(q, \lambda_i)}. \quad (8.115)$$

Next, we consider the commutator with $\mathcal{T}_k^k + \mathcal{T}_{34,k-1}^{34,k-1}$. Upon dismissing terms that vanish because they have annihilation operators acting on the vacuum, we find

$$\begin{aligned} \left[\mathcal{T}_k^k + \mathcal{T}_{34,k-1}^{34,k-1} \right] B_3(\lambda) &= \frac{\mathcal{X}_k^{k,0}}{\mathcal{Y}_{k;1}^{k,0;1}} B_3(\lambda) \left[\mathcal{T}_k^k + \mathcal{T}_{34,k-1}^{34,k-1} \right] + \\ &\quad \frac{\mathcal{Y}_{k;2}^{k,1;1}}{\mathcal{Y}_{k;1}^{k,0;1}} \left\{ \mathcal{T}_{3,k}^k B - \mathcal{T}_{34,k-1}^{4,k-1} A_3^3 \right\} + \frac{\mathcal{Y}_{k;4}^{k,1;1}}{\mathcal{Y}_{k;1}^{k,0;1}} \left\{ \mathcal{T}_{3,k}^{34,k-1} B + \mathcal{T}_k^{4,k-1} A_3^3 \right\}. \end{aligned} \quad (8.116)$$

If we now define $\hat{\Phi}_{k;i}(q, \lambda) = \mathcal{T}_k^{4,k-1}(q) \prod_{j \neq i} B_3(\lambda_j)$, $\hat{\Psi}_{k;i}(q, \lambda) = \mathcal{T}_{34,k-1}^{4,k-1}(q) \prod_{j \neq i} B_3(\lambda_j)$, then we can repeat the above steps to find

$$\begin{aligned} \left[\mathcal{T}_k^k + \mathcal{T}_{34,k-1}^{34,k-1} \right] \Phi(\lambda_1, \dots, \lambda_a) &= \prod_{i=1}^a \frac{\mathcal{X}_k^{k,0}(q, \lambda_i)}{\mathcal{Y}_{k;1}^{k,0;1}(q, \lambda_i)} \Phi(\lambda_1, \dots, \lambda_a) \left[\mathcal{T}_k^k + \mathcal{T}_{34,k-1}^{34,k-1} \right] + \\ &\quad \sum_{i=1}^a c_i P_{1i} \left\{ \Phi_{k;i}(q, \lambda) B(\lambda_i) - \hat{\Psi}_{k;i}(q, \lambda) A_3^3(\lambda_i) \right\} + \\ &\quad \sum_{i=1}^a d_i P_{1i} \left\{ \Psi_{k;i}(q, \lambda) B(\lambda_i) + \hat{\Phi}_{k;i}(q, \lambda) A_3^3(\lambda_i) \right\}. \end{aligned} \quad (8.117)$$

The last commutation relation finally gives

$$\begin{aligned} \mathcal{T}_{4,k}^{4,k}(q) \Phi(\lambda_1, \dots, \lambda_a) &= \frac{\mathcal{X}_{k+1}^{k+1,0}}{\mathcal{Y}_{k;1}^{k,0;1}} \frac{u_q - u_{\lambda_1} + \frac{\ell_0-1}{2} - k}{u_q - u_{\lambda_1} + \frac{\ell_0-3}{2} - k} B_3(\lambda_1) \mathcal{T}_{4,k}^{4,k}(q) \Phi(\lambda_2, \dots, \lambda_a) \\ &\quad - \frac{\mathcal{Y}_{k+1;2}^{k+1,1;1}}{\mathcal{Y}_{k+1;1}^{k+1,0;1}} \mathcal{T}_{34,k}^{4,k}(q) B(\lambda_1) \Phi(\lambda_2, \dots, \lambda_a) \\ &\quad + \frac{\mathcal{Y}_{k+1;4}^{k+1,1;1}}{\mathcal{Y}_{k+1;1}^{k+1,0;1}} \mathcal{T}_{k+1}^{4,k}(q) B(\lambda_1) \Phi(\lambda_2, \dots, \lambda_a). \end{aligned} \quad (8.118)$$

By summing all the terms, we find that $|a\rangle$ is indeed an eigenstate of the transfer matrix, provided that the parameters λ_i satisfy

$$B(\lambda_i)|0\rangle_P = A_3^3(\lambda_i)|0\rangle_P. \quad (8.119)$$

When working this out, we only find a dependence on $x^+(\lambda_i)$, which we denote as $y_i \equiv x^+(\lambda_i)$. The explicit formula is given by

$$\prod_{j=1}^{K^1} \frac{y_i - x_j^+}{y_i - x_j^-} \sqrt{\frac{x_j^-}{x_j^+}} = 1, \quad (8.120)$$

which agrees with the known auxiliary BAE for $K^{\text{III}} = 0$ from (8.56). The explicit eigenvalue of $|a\rangle$ is given by

$$\begin{aligned}
\Lambda(q|\vec{p}) &= \prod_{m=1}^{K^{\text{II}}} \frac{\mathcal{X}_0^{0,0}(q, \lambda_m)}{\mathcal{Y}_{0;1}^{0,0;1}(q, \lambda_m)} + \prod_{i=1}^{K^{\text{I}}} \mathcal{X}_{\ell_0;1}^{\ell_0,0;1}(q, p_i) \prod_{m=1}^{K^{\text{II}}} \frac{\mathcal{X}_{\ell_0}^{\ell_0,0}(q, \lambda_m)}{\mathcal{Y}_{\ell_0;1}^{\ell_0,0;1}(q, \lambda_m)} + \\
&\quad \sum_{k=1}^{\ell_0-1} \prod_{m=1}^{K^{\text{II}}} \frac{\mathcal{X}_k^{k,0}(q, \lambda_m)}{\mathcal{Y}_{k;1}^{k,0;1}(q, \lambda_m)} \left\{ \prod_{i=1}^{K^{\text{I}}} \lambda_+(q, p_i) + \prod_{i=1}^{K^{\text{I}}} \lambda_-(q, p_i) \right\} + \\
&\quad - \sum_{k=0}^{\ell_0-1} \prod_{m=1}^{K^{\text{II}}} \frac{\mathcal{X}_k^{k,0}(q, \lambda_m)}{\mathcal{Y}_{k;1}^{k,0;1}(q, \lambda_m)} \left[1 + \frac{u_q - u_{\lambda_m} + \frac{\ell_0}{2} - k}{u_q - u_{\lambda_m} + \frac{\ell_0-2}{2} - k} \right] \prod_{i=1}^{K^{\text{I}}} \mathcal{Y}_{k;1}^{k,0;1}(q, p_i).
\end{aligned} \tag{8.121}$$

This is indeed the case $K^{\text{III}} = 0$ of (8.56).

Bibliography

- [1] G. 't Hooft, *A planar diagram theory for strong interactions*, *Nucl. Phys.* **B72** (1974) 461.
- [2] J. Polchinski, *Dirichlet-Branes and Ramond-Ramond Charges*, *Phys. Rev. Lett.* **75** (1995) 4724–4727, [[hep-th/9510017](#)].
- [3] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231–252, [[hep-th/9711200](#)].
- [4] J. A. Minahan and K. Zarembo, *The Bethe-ansatz for $\mathcal{N} = 4$ super Yang-Mills*, *JHEP* **03** (2003) 013, [[hep-th/0212208](#)].
- [5] N. Beisert, C. Kristjansen, and M. Staudacher, *The dilatation operator of $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B664** (2003) 131–184, [[hep-th/0303060](#)].
- [6] N. Beisert and M. Staudacher, *The $N=4$ SYM Integrable Super Spin Chain*, *Nucl. Phys.* **B670** (2003) 439–463, [[hep-th/0307042](#)].
- [7] D. Serban and M. Staudacher, *Planar $N = 4$ gauge theory and the Inozemtsev long range spin chain*, *JHEP* **06** (2004) 001, [[hep-th/0401057](#)].
- [8] N. Beisert, *The dilatation operator of $N = 4$ super Yang-Mills theory and integrability*, *Phys. Rept.* **405** (2005) 1–202, [[hep-th/0407277](#)].
- [9] R. R. Metsaev and A. A. Tseytlin, *Type IIB superstring action in $AdS_5 \times S^5$ background*, *Nucl. Phys.* **B533** (1998) 109–126, [[hep-th/9805028](#)].
- [10] G. Arutyunov and S. Frolov, *Foundations of the $AdS_5 \times S^5$ Superstring. Part I*, [arXiv:0901.4937](#).
- [11] I. Bena, J. Polchinski, and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, *Phys. Rev.* **D69** (2004) 046002, [[hep-th/0305116](#)].

-
- [12] G. Arutyunov and M. Staudacher, *Matching higher conserved charges for strings and spins*, *JHEP* **03** (2004) 004, [[hep-th/0310182](#)].
 - [13] V. A. Kazakov, A. Marshakov, J. A. Minahan, and K. Zarembo, *Classical / quantum integrability in AdS/CFT*, *JHEP* **05** (2004) 024, [[hep-th/0402207](#)].
 - [14] S. Frolov and A. A. Tseytlin, *Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$* , *JHEP* **06** (2002) 007, [[hep-th/0204226](#)].
 - [15] S. Frolov and A. A. Tseytlin, *Multi-spin string solutions in $AdS_5 \times S^5$* , *Nucl. Phys.* **B668** (2003) 77–110, [[hep-th/0304255](#)].
 - [16] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, *Nucl. Phys.* **B636** (2002) 99–114, [[hep-th/0204051](#)].
 - [17] S. Frolov and A. A. Tseytlin, *Quantizing three-spin string solution in $AdS_5 \times S^5$* , *JHEP* **07** (2003) 016, [[hep-th/0306130](#)].
 - [18] S. Frolov and A. A. Tseytlin, *Rotating string solutions: AdS/CFT duality in non-supersymmetric sectors*, *Phys. Lett.* **B570** (2003) 96–104, [[hep-th/0306143](#)].
 - [19] G. Arutyunov, S. Frolov, J. Russo, and A. A. Tseytlin, *Spinning strings in $AdS_5 \times S^5$ and integrable systems*, *Nucl. Phys.* **B671** (2003) 3–50, [[hep-th/0307191](#)].
 - [20] G. Arutyunov, J. Russo, and A. A. Tseytlin, *Spinning strings in $AdS_5 \times S^5$: New integrable system relations*, *Phys. Rev.* **D69** (2004) 086009, [[hep-th/0311004](#)].
 - [21] S. A. Frolov, I. Y. Park, and A. A. Tseytlin, *On one-loop correction to energy of spinning strings in S^5* , *Phys. Rev.* **D71** (2005) 026006, [[hep-th/0408187](#)].
 - [22] I. Y. Park, A. Tirziu, and A. A. Tseytlin, *Spinning strings in $AdS_5 \times S^5$: One-loop correction to energy in $SL(2)$ sector*, *JHEP* **03** (2005) 013, [[hep-th/0501203](#)].
 - [23] N. Beisert, J. A. Minahan, M. Staudacher, and K. Zarembo, *Stringing spins and spinning strings*, *JHEP* **09** (2003) 010, [[hep-th/0306139](#)].
 - [24] N. Beisert, S. Frolov, M. Staudacher, and A. A. Tseytlin, *Precision spectroscopy of AdS/CFT*, *JHEP* **10** (2003) 037, [[hep-th/0308117](#)].
 - [25] N. Beisert, A. A. Tseytlin, and K. Zarembo, *Matching quantum strings to quantum spins: One-loop vs. finite-size corrections*, *Nucl. Phys.* **B715** (2005) 190–210, [[hep-th/0502173](#)].

-
- [26] R. Hernandez, E. Lopez, A. Perianez, and G. Sierra, *Finite size effects in ferromagnetic spin chains and quantum corrections to classical strings*, *JHEP* **06** (2005) 011, [[hep-th/0502188](#)].
- [27] N. Beisert and L. Freyhult, *Fluctuations and energy shifts in the Bethe ansatz*, *Phys. Lett.* **B622** (2005) 343–348, [[hep-th/0506243](#)].
- [28] N. Beisert and A. A. Tseytlin, *On quantum corrections to spinning strings and Bethe equations*, *Phys. Lett.* **B629** (2005) 102–110, [[hep-th/0509084](#)].
- [29] L. Freyhult and C. Kristjansen, *A universality test of the quantum string bethe ansatz*, *Phys. Lett.* **B638** (2006) 258–264, [[hep-th/0604069](#)].
- [30] M. Beccaria, G. V. Dunne, V. Forini, M. Pawellek, and A. A. Tseytlin, *Exact computation of one-loop correction to energy of spinning folded string in $AdS_5 \times S^5$* , [arXiv:1001.4018](#).
- [31] T. Klose, T. McLoughlin, R. Roiban, and K. Zarembo, *Worldsheet scattering in $AdS_5 \times S^5$* , *JHEP* **03** (2007) 094, [[hep-th/0611169](#)].
- [32] V. Giangreco Marotta Puletti, T. Klose, and O. Ohlsson Sax, *Factorized world-sheet scattering in near-flat $AdS_5 \times S^5$* , *Nucl. Phys.* **B792** (2008) 228–256, [[arXiv:0707.2082](#)].
- [33] T. Klose, T. McLoughlin, J. A. Minahan, and K. Zarembo, *World-sheet scattering in $AdS_5 \times S^5$ at two loops*, *JHEP* **08** (2007) 051, [[arXiv:0704.3891](#)].
- [34] M. Staudacher, *The factorized S-matrix of CFT/AdS*, *JHEP* **05** (2005) 054, [[hep-th/0412188](#)].
- [35] G. Arutyunov and S. Frolov, *Integrable Hamiltonian for classical strings on $AdS_5 \times S^5$* , *JHEP* **02** (2005) 059, [[hep-th/0411089](#)].
- [36] S. Frolov, J. Plefka, and M. Zamaklar, *The $AdS_5 \times S^5$ superstring in light-cone gauge and its Bethe equations*, *J. Phys.* **A39** (2006) 13037–13082, [[hep-th/0603008](#)].
- [37] G. Arutyunov, S. Frolov, and M. Zamaklar, *Finite-size effects from giant magnons*, *Nucl. Phys.* **B778** (2007) 1–35, [[hep-th/0606126](#)].
- [38] N. Beisert, *The $su(2|2)$ dynamic S-matrix*, *Adv. Theor. Math. Phys.* **12** (2008) 945, [[hep-th/0511082](#)].
- [39] G. Arutyunov, S. Frolov, J. Plefka, and M. Zamaklar, *The off-shell symmetry algebra of the light-cone $AdS_5 \times S^5$ superstring*, *J. Phys.* **A40** (2007) 3583–3606, [[hep-th/0609157](#)].

-
- [40] G. Arutyunov, S. Frolov, and M. Zamaklar, *The Zamolodchikov-Faddeev algebra for $AdS_5 \times S^5$ superstring*, *JHEP* **04** (2007) 002, [[hep-th/0612229](#)].
- [41] R. A. Janik, *The $AdS_5 \times S^5$ superstring worldsheet S -matrix and crossing symmetry*, *Phys. Rev.* **D73** (2006) 086006, [[hep-th/0603038](#)].
- [42] G. Arutyunov, S. Frolov, and M. Staudacher, *Bethe ansatz for quantum strings*, *JHEP* **10** (2004) 016, [[hep-th/0406256](#)].
- [43] N. Beisert, R. Hernandez, and E. Lopez, *A crossing-symmetric phase for $AdS_5 \times S^5$ strings*, *JHEP* **11** (2006) 070, [[hep-th/0609044](#)].
- [44] N. Beisert, B. Eden, and M. Staudacher, *Transcendentality and crossing*, *J. Stat. Mech.* **0701** (2007) P021, [[hep-th/0610251](#)].
- [45] H. Bethe, *On the theory of metals. 1. eigenvalues and eigenfunctions for the linear atomic chain*, *Z. Phys.* **71** (1931) 205–226.
- [46] N. Beisert, V. Dippel, and M. Staudacher, *A novel long range spin chain and planar $\mathcal{N} = 4$ super Yang-Mills*, *JHEP* **07** (2004) 075, [[hep-th/0405001](#)].
- [47] N. Beisert and M. Staudacher, *Long-range $\mathfrak{psu}(2,2|4)$ Bethe ansatz for gauge theory and strings*, *Nucl. Phys.* **B727** (2005) 1–62, [[hep-th/0504190](#)].
- [48] M. J. Martins and C. S. Melo, *The Bethe ansatz approach for factorizable centrally extended S -matrices*, *Nucl. Phys.* **B785** (2007) 246–262, [[hep-th/0703086](#)].
- [49] M. de Leeuw, *Coordinate Bethe Ansatz for the String S -Matrix*, *J. Phys.* **A40** (2007) 14413–14432, [[0705.2369](#)].
- [50] L. D. Faddeev, *How algebraic Bethe ansatz works for integrable model*, [hep-th/9605187](#).
- [51] P. Dorey, *Exact S matrices*, [hep-th/9810026](#).
- [52] N. Dorey, *Magnon bound states and the AdS/CFT correspondence*, *J. Phys.* **A39** (2006) 13119–13128, [[hep-th/0604175](#)].
- [53] H.-Y. Chen, N. Dorey, and K. Okamura, *On the scattering of magnon boundstates*, *JHEP* **11** (2006) 035, [[hep-th/0608047](#)].
- [54] R. Roiban, *Magnon bound-state scattering in gauge and string theory*, *JHEP* **04** (2007) 048, [[hep-th/0608049](#)].

- [55] N. Beisert, *The Analytic Bethe Ansatz for a Chain with Centrally Extended $su(2|2)$ Symmetry*, *J. Stat. Mech.* **0701** (2007) P017, [[nlin/0610017](#)].
- [56] G. Arutyunov and S. Frolov, *On String S-matrix, Bound States and TBA*, *JHEP* **12** (2007) 024, [[0710.1568](#)].
- [57] M. Luscher, *Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories. 1. Stable Particle States*, *Commun. Math. Phys.* **104** (1986) 177.
- [58] C. N. Yang and C. P. Yang, *Thermodynamics of a One-Dimensional System of Bosons with Repulsive Delta-Function Interaction*, *Journal of Math. Phys.* **10** (July, 1969) 1115–1122.
- [59] A. B. Zamolodchikov, *Thermodynamic Bethe Ansatz in Relativistic Models. Scaling Three State Potts and Lee-Yang Models*, *Nucl. Phys.* **B342** (1990) 695–720.
- [60] J. Ambjorn, R. A. Janik, and C. Kristjansen, *Wrapping interactions and a new source of corrections to the spin-chain / string duality*, *Nucl. Phys.* **B736** (2006) 288–301, [[hep-th/0510171](#)].
- [61] R. A. Janik and T. Lukowski, *Wrapping interactions at strong coupling – the giant magnon*, *Phys. Rev.* **D76** (2007) 126008, [[0708.2208](#)].
- [62] M. P. Heller, R. A. Janik, and T. Lukowski, *A new derivation of Luscher F-term and fluctuations around the giant magnon*, *JHEP* **06** (2008) 036, [[arXiv:0801.4463](#)].
- [63] Z. Bajnok and R. A. Janik, *Four-loop perturbative Konishi from strings and finite size effects for multiparticle states*, *Nucl. Phys.* **B807** (2009) 625–650, [[arXiv:0807.0399](#)].
- [64] Z. Bajnok, R. A. Janik, and T. Lukowski, *Four loop twist two, BFKL, wrapping and strings*, *Nucl. Phys.* **B816** (2009) 376–398, [[arXiv:0811.4448](#)].
- [65] Z. Bajnok, A. Hegedus, R. A. Janik, and T. Lukowski, *Five loop Konishi from AdS/CFT*, [arXiv:0906.4062](#).
- [66] F. Fiamberti, A. Santambrogio, C. Sieg, and D. Zanon, *Wrapping at four loops in $N=4$ SYM*, *Phys. Lett.* **B666** (2008) 100–105, [[arXiv:0712.3522](#)].
- [67] V. N. Velizhanin, *The Four-Loop Konishi in $N=4$ SYM*, [arXiv:0808.3832](#).
- [68] F. Fiamberti, *Wrapping effects in supersymmetric gauge theories*, [arXiv:1003.3118](#).
- [69] G. Arutyunov and S. Frolov, *String hypothesis for the $AdS_5 \times S^5$ mirror*, *JHEP* **03** (2009) 152, [[arXiv:0901.1417](#)].

-
- [70] N. Gromov, V. Kazakov, A. Kozak, and P. Vieira, *Exact Spectrum of Anomalous Dimensions of Planar $N = 4$ Supersymmetric Yang-Mills Theory: TBA and excited states*, *Lett. Math. Phys.* **91** (2010) 265–287, [[arXiv:0902.4458](#)].
- [71] G. Arutyunov and S. Frolov, *Thermodynamic Bethe Ansatz for the $\text{AdS}_5 \times \text{S}^5$ Mirror Model*, *JHEP* **05** (2009) 068, [[arXiv:0903.0141](#)].
- [72] D. Bombardelli, D. Fioravanti, and R. Tateo, *Thermodynamic Bethe Ansatz for planar AdS/CFT : a proposal*, [arXiv:0902.3930](#).
- [73] N. Gromov, V. Kazakov, and P. Vieira, *Exact Spectrum of Anomalous Dimensions of Planar $N=4$ Supersymmetric Yang-Mills Theory*, *Phys. Rev. Lett.* **103** (2009) 131601, [[arXiv:0901.3753](#)].
- [74] N. Gromov, V. Kazakov, and P. Vieira, *Exact Spectrum of Planar $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory: Konishi Dimension at Any Coupling*, *Phys. Rev. Lett.* **104** (2010) 211601, [[arXiv:0906.4240](#)].
- [75] S. Frolov and R. Suzuki, *Temperature quantization from the TBA equations*, *Phys. Lett. B* **679** (2009) 60–64, [[arXiv:0906.0499](#)].
- [76] G. Arutyunov, S. Frolov, and R. Suzuki, *Exploring the mirror TBA*, *JHEP* **05** (2010) 031, [[arXiv:0911.2224](#)].
- [77] G. Arutyunov and S. Frolov, *Simplified TBA equations of the $\text{AdS}_5 \times \text{S}^5$ mirror model*, *JHEP* **11** (2009) 019, [[arXiv:0907.2647](#)].
- [78] A. Cavaglia, D. Fioravanti, and R. Tateo, *Extended Y -system for the $\text{AdS}_5/\text{CFT}_4$ correspondence*, [arXiv:1005.3016](#).
- [79] S. Frolov, *Konishi operator at intermediate coupling*, [arXiv:1006.5032](#).
- [80] G. Arutyunov, S. Frolov, and R. Suzuki, *Five-loop Konishi from the Mirror TBA*, *JHEP* **04** (2010) 069, [[arXiv:1002.1711](#)].
- [81] J. Balog and A. Hegedus, *5-loop Konishi from linearized TBA and the XXX magnet*, *JHEP* **06** (2010) 080, [[arXiv:1002.4142](#)].
- [82] J. Balog and A. Hegedus, *The Bajnok-Janik formula and wrapping corrections*, [arXiv:1003.4303](#).
- [83] N. Beisert, *The S -Matrix of AdS/CFT and Yangian Symmetry*, *PoS SOLVAY* (2006) 002, [[0704.0400](#)].

-
- [84] G. Arutyunov and S. Frolov, *The S-matrix of String Bound States*, *Nucl. Phys.* **B804** (2008) 90–143, [[arXiv:0803.4323](#)].
- [85] M. de Leeuw, *Bound States, Yangian Symmetry and Classical r-matrix for the $AdS_5 \times S^5$ Superstring*, *JHEP* **06** (2008) 085, [[arXiv:0804.1047](#)].
- [86] C. Ahn and R. I. Nepomechie, *Yangian symmetry and bound states in AdS/CFT boundary scattering*, *JHEP* **05** (2010) 016, [[arXiv:1003.3361](#)].
- [87] N. MacKay and V. Regelskis, *On the reflection of magnon bound states*, [arXiv:1006.4102](#).
- [88] G. Arutyunov, M. de Leeuw, and A. Torrielli, *The Bound State S-Matrix for $AdS_5 \times S^5$ Superstring*, *Nucl. Phys.* **B819** (2009) 319–350, [[arXiv:0902.0183](#)].
- [89] O. Lunin and J. M. Maldacena, *Deforming field theories with $U(1) \times U(1)$ global symmetry and their gravity duals*, *JHEP* **05** (2005) 033, [[hep-th/0502086](#)].
- [90] S. Frolov, *Lax pair for strings in Lunin-Maldacena background*, *JHEP* **05** (2005) 069, [[hep-th/0503201](#)].
- [91] S. A. Frolov, R. Roiban, and A. A. Tseytlin, *Gauge - string duality for superconformal deformations of $N = 4$ super Yang-Mills theory*, *JHEP* **07** (2005) 045, [[hep-th/0503192](#)].
- [92] N. Beisert and R. Roiban, *Beauty and the twist: The Bethe ansatz for twisted $N = 4$ SYM*, *JHEP* **08** (2005) 039, [[hep-th/0505187](#)].
- [93] L. F. Alday, G. Arutyunov, and S. Frolov, *Green-Schwarz strings in TsT-transformed backgrounds*, *JHEP* **06** (2006) 018, [[hep-th/0512253](#)].
- [94] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, *$N=6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **10** (2008) 091, [[arXiv:0806.1218](#)].
- [95] G. Arutyunov and S. Frolov, *Superstrings on $AdS_4 \times CP^3$ as a Coset Sigma-model*, *JHEP* **09** (2008) 129, [[arXiv:0806.4940](#)].
- [96] B. Stefanski, jr, *Green-Schwarz action for Type IIA strings on $AdS_4 \times CP^3$* , *Nucl. Phys.* **B808** (2009) 80–87, [[arXiv:0806.4948](#)].
- [97] N. Gromov and P. Vieira, *The all loop AdS_4/CFT_3 Bethe ansatz*, *JHEP* **01** (2009) 016, [[arXiv:0807.0777](#)].

-
- [98] D. Bombardelli, D. Fioravanti, and R. Tateo, *TBA and Y-system for planar AdS_4/CFT_3* , [arXiv:0912.4715](#).
- [99] N. Gromov and F. Levkovich-Maslyuk, *Y-system, TBA and Quasi-Classical strings in $AdS_4 \times CP^3$* , [arXiv:0912.4911](#).
- [100] S. Moriyama and A. Torrielli, *A Yangian Double for the AdS/CFT Classical r -matrix*, *JHEP* **06** (2007) 083, [[0706.0884](#)].
- [101] N. Beisert and F. Spill, *The Classical r -matrix of AdS/CFT and its Lie Bialgebra Structure*, *Commun. Math. Phys.* **285** (2009) 537–565, [[arXiv:0708.1762](#)].
- [102] M. de Leeuw, *The Bethe Ansatz for $AdS_5 \times S^5$ Bound States*, *JHEP* **01** (2009) 005, [[arXiv:0809.0783](#)].
- [103] O. Babelon, D. Bernard, and M. Talon, *Introduction to Classical Integrable Systems*, Cambridge University Press (2003).
- [104] A. B. Zamolodchikov and A. B. Zamolodchikov, *Factorized S -matrices in two dimensions as the exact solutions of certain relativistic quantum field models*, *Annals Phys.* **120** (1979) 253–291.
- [105] G. W. Delius, *Exact S matrices with affine quantum group symmetry*, *Nucl. Phys.* **B451** (1995) 445–468, [[hep-th/9503079](#)].
- [106] A. B. Zamolodchikov, *Exact Two Particle s Matrix of Quantum Sine-Gordon Solitons*, *Pisma Zh. Eksp. Teor. Fiz.* **25** (1977) 499–502.
- [107] V. G. Drinfeld, *Quasi-Hopf algebras*, *Leningrad Math. J.* **1** (1990) 1419.
- [108] V. Stukopin, *Yangians of classical lie superalgebras: Basic constructions, quantum double and universal R -matrix*, *Proceedings of the Institute of Mathematics of NAS of Ukraine* **50** (2004) 1195.
- [109] N. J. MacKay, *Introduction to Yangian symmetry in integrable field theory*, *Int. J. Mod. Phys.* **A20** (2005) 7189–7218, [[hep-th/0409183](#)].
- [110] A. Molev, *Yangians and Classical Lie Algebras*, *Mathematical Surveys and Monographs 143*, American Mathematical Society, Providence, RI (2007).
- [111] P. Etingof and O. Schiffman, *Lectures on Quantum Groups*, *Lectures in Mathematical Physics*, International Press, Boston (1998).

-
- [112] V. Chari and A. Pressley, *A Guide To Quantum Groups*, Cambridge, UK: Univ. Press (1994).
- [113] V. G. Drinfeld, *Quantum groups*, *Proc. of the International Congress of Mathematicians, Berkeley, 1986*, American Mathematical Society (1987) 798.
- [114] V. G. Drinfeld, *A new realization of Yangians and quantum affine algebras*, *Soviet Math. Dokl.* **36** (1988) 212.
- [115] L. D. Faddeev, *Quantum completely integral models of field theory*, *Sov. Sci. Rev.* **C1** (1980) 107–155.
- [116] S. M. Khoroshkin and V. N. Tolstoy, *Yangian double*, *Letters in Mathematical Physics* **36** (1995) 373–402, [[hep-th/9406194](#)].
- [117] J. Cai, S. Wang, K. Wu, and C. Xiong, *Universal R-matrix Of The Super Yangian Double $DY(gl(1|1))$* , *Comm. Theor. Phys.* **29** (1998) 173–176, [[q-alg/9709038](#)].
- [118] F. Spill, *Weakly coupled $N=4$ Super Yang-Mills and $N=6$ Chern-Simons theories from $u(2|2)$ Yangian symmetry*, [arXiv:0810.3897](#).
- [119] D. Arnaudon, N. Crampe, L. Frappat, and É. Ragoucy, *Super yangian $y(osp(1|2))$ and the universal r-matrix of its quantum double*, *Commun. Math. Phys.* **240** (2003) 31.
- [120] F. H. L. Essler, H. Frahm, F. Goehmann, A. Kluemper, and V. E. Korepin, *The One-Dimensional Hubbard Model*, Cambridge University Press (2005).
- [121] D. Bernard, *An Introduction to Yangian Symmetries*, *Int. J. Mod. Phys.* **B7** (1993) 3517–3530, [[hep-th/9211133](#)].
- [122] N. Sochen, *Integrable generalized principal chiral models*, *Phys. Lett.* **B391** (1997) 374–380, [[hep-th/9607009](#)].
- [123] Y.-Z. Zhang and M. D. Gould, *A Unified and Complete Construction of All Finite Dimensional Irreducible Representations of $\mathfrak{gl}(2|2)$* , *J. Math. Phys.* **46** (2005) 013505, [[math/0405043](#)].
- [124] G. Gotz, T. Quella, and V. Schomerus, *Tensor products of $psl(2|2)$ representations*, [hep-th/0506072](#).
- [125] C. Gomez and R. Hernandez, *The magnon kinematics of the AdS/CFT correspondence*, *JHEP* **11** (2006) 021, [[hep-th/0608029](#)].

-
- [126] J. Plefka, F. Spill, and A. Torrielli, *On the Hopf algebra structure of the AdS/CFT S-matrix*, *Phys. Rev.* **D74** (2006) 066008, [[hep-th/0608038](#)].
- [127] T. Matsumoto and S. Moriyama, *An Exceptional Algebraic Origin of the AdS/CFT Yangian Symmetry*, *JHEP* **04** (2008) 022, [[arXiv:0803.1212](#)].
- [128] F. Spill and A. Torrielli, *On Drinfeld's second realization of the AdS/CFT $su(2|2)$ Yangian*, *J. Geom. Phys.* (in press) [[arXiv:0803.3194](#)].
- [129] H.-Y. Chen, N. Dorey, and K. Okamura, *The asymptotic spectrum of the $N = 4$ super Yang-Mills spin chain*, *JHEP* **03** (2007) 005, [[hep-th/0610295](#)].
- [130] I. Heckenberger, F. Spill, A. Torrielli, and H. Yamane, *Drinfeld second realization of the quantum affine superalgebras of $D^{(1)}(2,1:x)$ via the Weyl groupoid*, *RIMS Kokyuroku Bessatsu* **B8** (2008) 171–216, [[arXiv:0705.1071](#)].
- [131] J. Van Der Jeugt, *Irreducible representations of the exceptional Lie superalgebras $D(2,1;\alpha)$* , *J. Math. Phys.* **26** (1985) 913–924.
- [132] L. Frappat, P. Sorba, and A. Sciarrino, *Dictionary on Lie superalgebras*, [hep-th/9607161](#).
- [133] R. Hernandez and E. Lopez, *Quantum corrections to the string bethe ansatz*, *JHEP* **07** (2006) 004, [[hep-th/0603204](#)].
- [134] N. Beisert, T. McLoughlin, and R. Roiban, *The Four-Loop Dressing Phase of $N=4$ SYM*, *Phys. Rev.* **D76** (2007) 046002, [[arXiv:0705.0321](#)].
- [135] B. Eden and M. Staudacher, *Integrability and transcendentality*, *J. Stat. Mech.* **0611** (2006) P014, [[hep-th/0603157](#)].
- [136] D. A. Varshalovich, A. N. Moksalev, and V. K. Khersonskii, *Quantum Theory of Angular Momentum*. World Scientific Publishing Co., 1988.
- [137] A. A. Belavin and V. G. Drinfel'd, *Solutions of the classical Yang - Baxter equation for simple Lie algebras*, *JHEP* **16** (1982) 159–180.
- [138] R.-b. Zhang, M. D. Gould, and A. J. Bracken, *Solutions of the graded classical Yang-Baxter equation and integrable models*, *J. Phys.* **A24** (1991) 1185–1198.
- [139] D. A. Leites and V. V. Serganova, *Solutions of the classical Yang-Baxter equation for simple superalgebras*, *Theor. Math. Phys.* **58** (1984) 16–24.

-
- [140] G. Karaali, *Constructing r -matrices on simple lie superalgebras*, *Journal of Algebra* **282** (2004), no. 1 83 – 102.
- [141] A. Torrielli, *Classical r -matrix of the $su(2|2)$ SYM spin-chain*, *Phys. Rev.* **D75** (2007) 105020, [[hep-th/0701281](#)].
- [142] G. Arutyunov and S. Frolov, *On $AdS_5 \times S^5$ string S -matrix*, *Phys. Lett.* **B639** (2006) 378–382, [[hep-th/0604043](#)].
- [143] D. E. Berenstein, J. M. Maldacena, and H. S. Nastase, *Strings in flat space and pp waves from $N = 4$ super Yang Mills*, *JHEP* **04** (2002) 013, [[hep-th/0202021](#)].
- [144] N. Beisert, *An $SU(1|1)$ -invariant S -matrix with dynamic representations*, *Bulg. J. Phys.* **33S1** (2006) 371–381, [[hep-th/0511013](#)].
- [145] S. M. Khoroshkin and V. N. Tolstoy, *Universal R -matrix for quantized (super)algebras*, *Commun. Math. Phys.* **276** (1991) 599–617.
- [146] A. Torrielli, *Structure of the string R -matrix*, *J. Phys.* **A42** (2009) 055204, [[arXiv:0806.1299](#)].
- [147] C.-N. Yang, *Some exact results for the many body problems in one dimension with repulsive delta function interaction*, *Phys. Rev. Lett.* **19** (1967) 1312–1314.
- [148] H. B. Thacker, *Exact integrability in quantum field theory and statistical systems*, *Rev. Mod. Phys.* **53** (Apr, 1981) 253–285.
- [149] N. M. Bogoliubov, A. G. Izergin, and V. E. Korepin, *Quantum Inverse Scattering Method and Correlation Functions*, Cambridge University Press (1993).
- [150] P. B. Ramos and M. J. Martins, *The quantum inverse scattering method for Hubbard-like models*, *Nucl. Phys.* **B522** (1998) 413–470, [[solv-int/9712014](#)].
- [151] P. B. Ramos and M. J. Martins, *Algebraic Bethe ansatz approach for the one-dimensional Hubbard model*, *J. Phys.* **A30** (1997) L195–L202, [[hep-th/9605141](#)].
- [152] N. Beisert, V. A. Kazakov, K. Sakai, and K. Zarembo, *Complete spectrum of long operators in $N = 4$ SYM at one loop*, *JHEP* **07** (2005) 030, [[hep-th/0503200](#)].
- [153] Y. Hatsuda and R. Suzuki, *Finite-Size Effects for Dyonic Giant Magnons*, *Nucl. Phys.* **B800** (2008) 349–383, [[arXiv:0801.0747](#)].
- [154] G. Arutyunov, M. de Leeuw, R. Suzuki, and A. Torrielli, *Bound State Transfer Matrix for $AdS_5 \times S^5$ Superstring*, *JHEP* **10** (2009) 025, [[arXiv:0906.4783](#)].